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Research Paper

The Poisson-Weibull Regression Model

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Abstract

In this paper, we proposed a regression model in the form location-scale based on the Poisson-Weibull distribution. This distribution arises on the competitive risk scenario. The new regression model was proposed for modeling data which has a increasing, decreasing and unimodal failure rate, and presented as particular cases the new exponential-Poisson regression model and the Weibull regression model. Assuming censored data, we considered the maximum likelihood approach for parameters estimation. For different parameter values, sample sizes and censoring percentages, various simulation studies were performed to study the means, bias relative and mean square error of the maximum likelihood estimative, and to compare the performance of the Poisson-Weibull regression model and their particular cases. The selection criteria AIC and likelihood ratio test were used for selection of regression model. Besides, we used the sensitivity analysis to detect influential or outlying observations and residual analysis was used to check assumptions in the model. The relevance of the approach was illustrated with a data set.

Keywords: Poisson-Weibull Distribution · Censored Data · Regression Model · Residual Analysis · Sensitivity Analysis.

1. Introduction

The Weibull distribution has been widely used for analysis of survival data in medical and engineering applications. However, the Weibull distribution does not provide a reasonable parametric fit for phenomenon modeling with non-monotone failure rates, such as the bathtub-shaped and unimodal failure rates, which are common in reliability and biological studies. In the last decade, new classes of models have been proposed for data modeling of this kind, based on extended forms of the Weibull distribution.

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Besides, in many applications it is not possible to observe the exact value of survival time, but it can only be observed the minimum or maximum value of this time. This occurs, for instance, when the interest is to observe the lifetime of a system in series or in parallel. In this case, the lifetime duration depends on a set of components. On the other hand, in a context of competing risks, there is no information about which the risk was responsible for the component failure and only the minimum lifetime value among all risks is observed. In recent years, several authors proposed probability distributions which properly accommodate survival data in presence of latent competing risks. For example, Adamidis and Loukas (1998) that propose a compounding distribution, denoted by exponential geometric (EG) distribution; Kus (2007) proposed the exponential-Poisson distribution (EP); and Louzada et al. (2011b) proposed the complementary exponential geometric distribution. There are others works, such as Tahmasbi and Rezaei (2008), who introduced the logarithmic exponential distributions, and Chahkandi and Ganjali (2009), introduced the exponential power series (EPS), which contains the distributions cited (EG, EP, and logarithmic exponential) as special cases.

Additionally, lifetimes are affected by variables, which are referred as explanatory variables or covariates. In industry, for example, the survival time of a given device can be influenced by the voltage level at which the equipment is subjected. In this case, an approach based on a regression model can be used. There are two classes of regression models proposed in the literature: parametric and semiparametric models. More details in Cox and Oakes (1984), Kalbfleisch and Prentice (2002), Lawless (2003), among others. In this paper, we shall be concerned only with parametric forms. So, a location-scale regression model based on the Poisson-Weibull distribution (Louzada et al., 2011a), denoted as Poisson-Weibull regression model, is proposed as an alternative for data modeling with a increasing, decreasing and unimodal failure rate function. This distribution arises on a latent complementary risk problem base and is obtained by compounding of the Weibull and Poisson distributions. Some regression models have been proposed with this objective, among them: log-Burr XII regression model (Silva et al., 2008); log-extended Weibull regression model (Silva et al., 2009); Kumaraswamy-logistic regression model (Santana et al., 2012); log-beta Weibull regression model (Kattan et al., 2013); log-Kumaraswamy generalized gamma regression model (Pascoa et al., 2013); log-McDonald Weibull regression model (Cordeiro et al., 2014); and extended Burr XII regression model (Lanjoni et al., 2016).

The new regression model, due to its flexibility in accommodating various forms of the risk function, seems to be an important model that can be used in a variety of problems in the survival data modeling. In addition to that, the Poisson-Weibull regression model is also suitable for testing goodness-of-fit of some special sub-models, such as the exponential-Poisson and Weibull regression models. We demonstrate by means of an application to real data that the Poisson-Weibull regression model can produce better fits than some known models. Hence, it represents a good alternative for lifetime data analysis, and we hope this generalization may attract wider applications in survival analysis. The inferential part of this model is carried out using the asymptotic distribution of the maximum likelihood estimators. Considering that the Poisson-Weibull model is embedded in the exponential-

Poisson and Weibull regression models, the likelihood ratio test can be used to discriminate such models. Studies were conducted via Monte Carlo simulation in order to evaluate the performance of the Poisson-Weibull regression model by means, bias relative and mean squared error for the maximum likelihood estimates ($MLEs$) and the size and power of the likelihood ratio test for model selection.

After modelling, it is important to check assumptions in the model and to conduct a robustness study to detect influential or extreme observation that can cause distortions to the results of the analysis. Numerous approaches have been proposed in the literature to detect influential or outlying observations. An efficient way to detect influential observations, proposed by Cook (1986), is the local influence approach, where one again investigates how the results of an analysis are changed under small perturbations in the model, and where these perturbations can be specific interpretations. Using this general method and also applying the method of Poon and Poon (1999), in this paper we develop a local influence approach for Poisson-Weibull regression models with censored data.

In survival analysis, there are various papers exploring new distributions, such as the Ortega et al. (2003) that considered the problem of assessing local influence in generalized log-gamma regression models with censored observations, Silva et al. (2008) that considered the problem of assessing local influence in log-Burr regression models with censored data, Ortega et al. (2010) who considered local influence for the generalized log-gamma regression models with cure fraction, Hashimoto et al. (2012) that discussed local influence for the log-Burr XII regression model for grouped survival data, Hashimoto et al. (2013) who investigated local influence in the new Neyman type A beta Weibull regression model. Recently Fachini et al. (2014) who considered local influence for the bivariate regression model with cure fraction and Ortega et al. (2014) derive the appropriate matrices for assessing local influence on the parameter estimates under different perturbation schemes for the odd Weibull regression model.

Another important step after the formulation of the model is the residuals analysis. Starting from this analysis, we can make evaluations if the proposed model is appropriate, to identify outliers and to observe whether there are differences in the assumptions made in the proposed model. In the survival analysis various residuals were proposed (see, for example, Collet, 1994). Additionally, the examination of Cox-Snell residual (Cox and Snell, 1968) was used to check assumptions in the model.

This paper was organized as follows. In the sections 2 and 3 we presented the Poisson-Weibull distribution and the Poisson-Weibull regression model. In the sections 4 and 5 we presented the inferential procedure based on maximum likelihood approach, the selection criteria AIC, BIC and the likelihood ratio test to select the best model. In the section 6 we presented the results of a simulation study conducted to assess the performance of the new regression model. In the section 7 we used several diagnostic measures considering three perturbation schemes in Poisson-Weibull regression model with censored observations. We also used the Cox-Snell residual to verify the goodness of fit in section 8. Finally, in section 9 the data set was analyzed and the final considerations appear in section 10.

2. The Poisson-Weibull Distribution

The Poisson-Weibull (PW) distribution inserted in the latent competitive risk scenario, where there is no information about which factor was responsible for the component failure, only the minimum lifetime value among all risks was observed. The PW density function (Louzada et al., 2011a) is given by,

$$f(t) = \frac{\alpha \exp\{\alpha \exp[-(\beta t)^{\gamma}] - (\beta t)^{\gamma}\} \beta^{\gamma} t^{\gamma-1}\gamma}{\exp(\alpha) - 1}, \tag{1}$$

where $t > 0$, $\beta > 0$ is scale parameter, $\alpha > 0$ and $\gamma > 0$ are shape parameters.

The survival and hazard functions corresponding to (1) are given by, respectively,

$$S(t) = \frac{\exp\{\alpha \exp[-(\beta t)^{\gamma}]\} - 1}{\exp(\alpha) - 1},$$

and

$$h(t) = \frac{\alpha \exp\{\alpha \exp[-(\beta t)^{\gamma}] - (\beta t)^{\gamma}\} \beta^{\gamma} t^{\gamma-1}\gamma}{\exp\{\alpha \exp[-(\beta t)^{\gamma}]\} - 1}. \tag{2}$$

Figure 1 illustrates some of the possible shapes of the hazard function (2) for selected parameter values. We noted from this figure that the hazard function is quite flexible and can accommodate various forms, such as increasing, decreasing and unimodal. Applications of the PW distribution in survival studies were investigated by Louzada et al. (2011a).

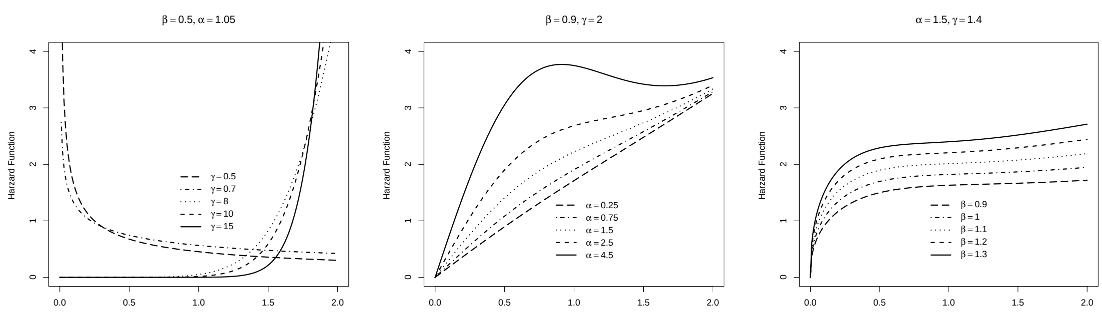


Figure 1. Plots of the failure rate function for Poisson-Weibull distribution.

Now, we define the random variable $Y = \log(T)$, log Poisson-Weibull distribution, parameterized in terms of $\gamma = 1/\sigma$ and $\beta = \exp(-\mu)$ by the density function

$$\begin{aligned} f(y) =& \frac{\alpha}{(\exp(\alpha)-1)\sigma} \exp\left\{\alpha \exp\left[-\left(\exp\left(\frac{y-\mu}{\sigma}\right)\right)\right] + \left(\frac{y-\mu}{\sigma}\right)\right\} \times \\ & \times \exp\left\{-\left[\exp\left(\frac{y-\mu}{\sigma}\right)\right]\right\}, \end{aligned} \tag{3}$$

where $-\infty < y < \infty$, $\alpha > 0$ is shape parameter, $\sigma > 0$ is scale parameter and $-\infty < \mu < \infty$ is location parameter. In addition, it can be viewed as a location-scale model for

$Y = \log(T)$. Plots of the density function (3) for some parameter values are given in Figure 2.

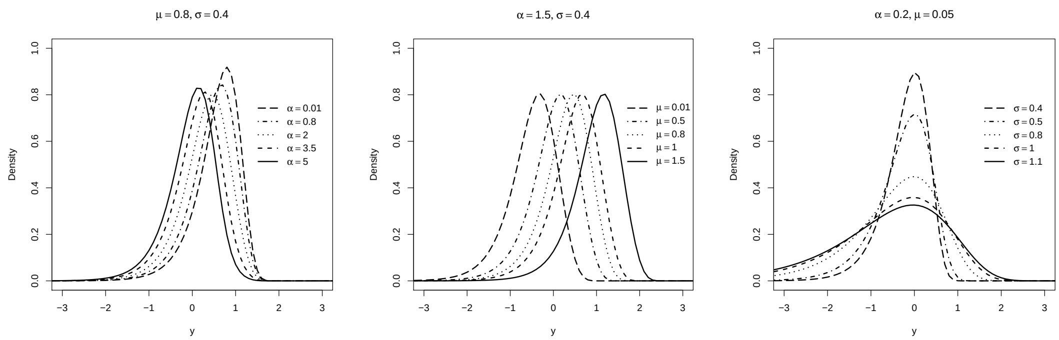


Figure 2. Plot of the density function for log Poisson-Weibull distribution.

The survival function of Y is given by

$$S(y) = \frac{\exp\left\{\alpha \exp\left[-\left(\exp\left(\frac{y-\mu}{\sigma}\right)\right)\right]\right\} - 1}{\exp(\alpha) - 1}.$$

3. The Poisson-Weibull Regression Model

In many pratical applications, lifetimes are affected by explanatory variables such as voltage, temperature and many others. The explanatory variable vector is denoted by $\mathbf{x} = (x_1, x_2, ..., x_p)^T$ which is related to response $Y = \log(T)$ through a regression model.

The location-scale regression model is given by

$$Y = \log(T) = \mu + \sigma Z, \tag{4}$$

where Y follows the distribution in (3). Hence, Z has the density function

$$f(z) = \frac{\alpha \exp\left\{z + \alpha \exp\left[-\exp(z)\right] + \exp\left[-\exp\left(z\right)\right]\right\}}{\exp(\alpha) - 1}, \quad -\infty < z < \infty. \tag{5}$$

Thus, the regression model based on the PW distribution (1) relating to the response Y and the covariate vector $\mathbf{x}$, can be expressed as

$$y_i = \mathbf{x}_i^T \boldsymbol{\beta} + \sigma z_i, \quad i = 1, \ldots, n, \tag{6}$$

where $\boldsymbol{\beta} = (\beta_0, \beta_1, \ldots, \beta_p)^T$ and $\sigma > 0$ are unknown parameters, $\mathbf{x}_i^T = (x_{i0}, x_{i1}, \ldots, x_{ip})$ is the explanatory vector and the random error z_i follows the density function (5). Hence, the linear predictor vector $\boldsymbol{\mu} = (\mu_1, \mu_2, \ldots, \mu_n)^T$ of the PW regression model is simply $\boldsymbol{\mu} = \mathbf{X}\boldsymbol{\beta}$, where $\mathbf{X} = (\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n)^T$ is a known model matrix.

In this case, the survival function of $Y|\mathbf{x}$ is given by

$$S(y|\mathbf{x}) = \frac{\exp\{\alpha \exp[-\exp((y - \mathbf{x}_i^T \boldsymbol{\beta})/\sigma))]\} - 1}{\exp(\alpha) - 1}. \tag{7}$$

The Poisson-Weibull regression model (6) opens new possibilities for several types of fitted data. It is observed that when $\alpha \to 0$ in equation (7) the Poisson-Weibull regression model is reduced to the Weibull regression model. For $\sigma = 1$ in equation (7) the Poisson-Weibull regression model is reduced to new exponential-Poisson regression model.

For the interpretation of the estimated coefficients, a possible proposal is based on the ratio of median times (Hosmer and Lemeshow, 1999). Therefore, when the covariable is binary (1 or 0), and considering the ratio of median times with $x = 1$ in the numerator, if $\hat{\beta}$ is negative (positive), it implies that the individuals with $x = 1$ decrease (increase) the median survival time in $\exp(\hat{\beta}) \times 100\%$ as compared to those individuals in the group with $x = 0$ by fixing the other covariables. This interpretation can be extended to continuous or categorical covariables.

4. Estimation by maximum likelihood

Given a random sample of size n composed by $(y_1, \mathbf{x}_1), (y_2, \mathbf{x}_2), \ldots, (y_n, \mathbf{x}_n)$, where y_i is the logarithm of the survival time that has distribution (3), and $\mathbf{x}_i$ is the covariate vector associated with the i_{th} individual. The log-likelihood function of the parameter vector $\boldsymbol{\theta} = (\alpha, \sigma, \boldsymbol{\beta}^T)^T$ can be written as

$$
\begin{aligned}
l(\boldsymbol{\theta}) \propto & \sum_{i=1}^{n} \delta_i \ln(\alpha) - \sum_{i=1}^{n} \delta_i \exp\left(\frac{y_i - \mathbf{x}_i^T \boldsymbol{\beta}}{\sigma}\right) + \sum_{i=1}^{n} \delta_i \left[\alpha \exp\left\{- \exp\left(\frac{y_i - \mathbf{x}_i^T \boldsymbol{\beta}}{\sigma}\right)\right\} + \frac{y_i - \mathbf{x}_i^T \boldsymbol{\beta}}{\sigma}\right] \\
& - \sum_{i=1}^{n} \delta_i \ln\left[\left(\exp(\alpha) - 1\right)\sigma\right] - \sum_{i=1}^{n} (1 - \delta_i) \ln\left[\exp\left\{\alpha \exp\left(- \exp\left(\frac{y_i - \mathbf{x}_i^T \boldsymbol{\beta}}{\sigma}\right)\right)\right\} - 1\right] \\
& - \sum_{i=1}^{n} (1 - \delta_i) \ln(\exp(\alpha) - 1), \qquad (8)
\end{aligned}
$$

where δ_i is equal to 0 or 1, respectively, if the data is censured or observed.

Maximum likelihood estimates ($MLEs$) for parameter vector $\boldsymbol{\theta} = (\alpha, \sigma, \boldsymbol{\beta}^T)^T$ can be obtained by maximizing the log-likelihood function (8) solving the system of equations given by

$$
U(\boldsymbol{\theta}) = \frac{\partial l(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} = \mathbf{0}.
$$

The components of the score vector $U(\boldsymbol{\theta})$ are given by

$$
\begin{aligned}
\frac{\partial l(\boldsymbol{\theta})}{\partial \alpha} = & \frac{1}{\alpha} \sum_{i=1}^{n} \delta_i + \sum_{i=1}^{n} \delta_i \exp\left\{- \exp\left(u_i\right)\right\} - \sum_{i=1}^{n} \delta_i \frac{\exp(\alpha)\sigma}{(\exp(\alpha) - 1)\sigma} \\
& - \sum_{i=1}^{n} (1 - \delta_i) \frac{\exp\left[\alpha \exp\left\{- \exp\left(u_i\right)\right\}\right] \exp\left\{- \exp\left(u_i\right)\right\}}{\exp\left[\alpha \exp\left\{- \exp\left(u_i\right)\right\}\right] - 1} - \sum_{i=1}^{n} (1 - \delta_i) \frac{\exp(\alpha)}{\exp(\alpha) - 1},
\end{aligned}
$$

$$\frac{\partial l(\boldsymbol{\theta})}{\partial \sigma} = -\sum_{i=1}^{n}(\delta_i)\exp(u_i)\,(-v_i) + \sum_{i=1}^{n}(\delta_i)\left[\alpha \exp\{-\exp(u_i)\} - \exp(u_i)\,(-v_i) - v_i\right]$$
$$- \sum_{i=1}^{n}(\delta_i)\frac{\exp(\alpha)-1}{(\exp(\alpha)-1)\sigma} + \frac{\exp\left[\alpha\exp\left(u_i\right)\right]\alpha\exp(u_i)(-v_i)}{\exp\left[\alpha\exp\left(u_i\right)\right]-1},$$
$$\frac{\partial l(\boldsymbol{\theta})}{\partial \beta_j} = -\sum_{i=1}^{n}(\delta_i)\exp(u_i)\left(-\frac{x_{ij}}{\sigma}\right) + \sum_{i=1}^{n}(\delta_i)\left[\alpha\exp\{-\exp\left(u_i\right)\}\exp(u_i)\left(-\frac{x_{ij}}{\sigma}\right)\left(-\frac{x_{ij}}{\sigma}\right)\right]$$
$$-\sum_{i=1}^{n}(1-\delta_i)\frac{\exp\left[\alpha\exp\{-\exp\left(u_i\right)\}\right]\alpha\exp\{-\exp\left(u_i\right)\} - \exp\left(u_i\right)\left(-\frac{x_{ij}}{\sigma}\right)}{\exp\left[\alpha\exp\{-\exp\left(u_i\right)\}\right]-1},$$

where $j = 0, 1, \ldots, p$, $u_i = (y_i - \mathbf{x}_i^T\boldsymbol{\beta})/\sigma$ and $v_i = (y_i - \mathbf{x}_i^T\boldsymbol{\beta})/\sigma^2$.

Since there is no closed analytical way to find these estimators, we can use numerical methods for solving the system of equations. Thus, estimates of these parameters were obtained by numerical methods, using an iterative process. We used the command *optim* in software *R* through of the method *BFGS*. In the case where the sample size is large and under certain conditions regularly to the likelihood function, confidence intervals and hypothesis testing for the parameters can be obtained using the fact that the maximum likelihood estimators, $\hat{\boldsymbol{\theta}}$, have asymptotic multivariate normal distribution with mean $\boldsymbol{\theta}$ and variance and covariance matrix $\boldsymbol{\Sigma}$, estimated by $I^{-1}(\boldsymbol{\theta}) = -E[\ddot{L}(\boldsymbol{\theta})]$, where $\ddot{L}(\boldsymbol{\theta}) = \partial^2 l(\boldsymbol{\theta})/\partial\boldsymbol{\theta}\partial\boldsymbol{\theta}^T$, this is, $\sqrt{n}(\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}) \sim N_p(\mathbf{0}, \mathbf{I}^{-1}(\boldsymbol{\theta}))$. Whereas the calculation of the $\mathbf{I}(\boldsymbol{\theta})$ is not possible by the presence of censored observations, alternatively we can use the information matrix observed, -$\ddot{L}(\boldsymbol{\theta})$, assessed in $\boldsymbol{\theta} = \widehat{\boldsymbol{\theta}}$, which is a consistent estimator for $\boldsymbol{\Sigma}$ (Mudholkar et al., 1995).

5. Model Selection

For the selection of the model that best fits the data, it was used the *AIC* model selection criteria (Akaike's information criterion) and *BIC* (Bayesian information criterion), and also the likelihood ratio test. The *AIC* and *BIC* are defined by

$$AIC = -2\log(\hat{L}) + 2p; \qquad BIC = -2\log(\hat{L}) + 2\log(n),$$

where $\hat{L}$ is the maximized value of the likelihood function of the model, p is the number of parameters of the model and n is the size sample. The preferred model is the one with the smallest value on each criterion.

When testing embbed models, the likelihood ratio test (LR) can be used to discriminate such models. We can compute the maximum values of the unrestricted and restricted log-likelihoods to construct the LR statistics (ω_n) for testing some sub-models. This statistic converge to a chi-squared distribution with degrees of freedom equal to the difference between the numbers of parameters in the two models. However, for comparison of non-embbed survival model, under certain conditions of regularity, the distribution of the statistical likelihood ratio under H_0 is a mixture with a weights (0.5 and 0.5) of distribution χ^2 with a degree of freedom, with a discrete distribution and with mass concentrated in the value 0, this is, $P(\omega_n \leq w) = \frac{1}{2} + \frac{1}{2}P(\chi_1^2 \leq w)$. More details in Maller and Zhou (1995) and Cancho et al. (2011).

6. Simulation Study

To examine the performance of Poisson-Weibull regression model compared with the Weibull and exponential-Poisson regression models and assess the performance of $MLEs$ for the parameters of the new model, a simulation study was done for different values of n with 0%, 10% and 30% censored observations in each sample, generating 1000 random samples simulated with the support of the *Software R*.

For the analysis of Poisson-Weibull regression model with their particular cases, a study was done for different values of n (60, 130, 200, 300, 400 and 500) in which the survival time of T follow PW distribution with density function given by (1), where they were generated from the inverse transformation method considering the following reparametrization $\gamma = 1/\sigma$ and $\beta = \exp(-\mu)$. The time of censure C is a random variable exponential with parameter $\lambda = 0.5$. Besides, the logarithm of the survival time (Y) and of censure in the simulation were considered through $y_i = \min(\log(t_i), \log(c_i))$.

The parameter values were fixed in $\alpha = 3.5$, and 5; $\sigma = 0.80$ and 0.85 and for each analyzed sample, Poisson-Weibull regression model given by (4), exponential-Poisson and Weibull regression models were adjusted with $\mu_i = \beta_0 + \beta_1 x_i$ where x_i from the uniform distribution on the interval $(0, 1)$ with β_0 and β_1 fixed. Below we describe the process of this simulation:

1. Generate $u_i \sim U(0, 1)$;
2. Determine $t_i = F^{-1}(u_i) = \frac{[\log(\alpha) - \log(\log(\exp(\alpha)(1-u_i)+u_i))]^{\sigma}}{\exp(-\mu_i)}$;
3. Generate variable of censure c_i; $c_i \sim \exp(0.5)$;
4. Find $y_i = \min(\log(t_i), \log(c_i))$;
5. If $\log(t_i) < \log(c_i)$, then $\delta_i = 1$, otherwise, $\delta_i = 0$, to $i = 1, \ldots, n$.

For each combination of n, parameter values and censoring percentages, 10000 samples were generated each one being adjusted under the Poisson-Weibull regression model (6) with $\mu_i = \beta_0 + \beta_1 x_i$. For each fit, was done the likelihood-ratio test for the hypothesis $H_0 : \sigma = 1$ versus $H_1 : \sigma \neq 1$, which is equivalent to compare the Poisson-Weibull regression model with the exponential-Poisson regression model. All the statistics of the tests were compared with the critical value χ_1^2 at a significance level of 5%. The simulations were performed for different n and different parameters values to get the simulated sizes and powers for test. Also the proportion of times that the AIC and BIC selected the Poisson-Weibull regression model was calculated. It is also realized the procedure for the analysis of the Poisson-Weibull and Weibull regression models, but in this case the hypotheses for test were $H_0 : \alpha \to 0$ $v.s$ $H_1 : \alpha > 0$. The percentile of 95° of this distribution represented by $w_{0.95}$, is such that $\frac{1}{2} + \frac{1}{2}P(\chi_1^2 \leq w_{0.95}) = 0.95$, so that $w_{0.95} = 2.705543$. Therefore, it rejects H_0 at a significance level of 5% to $\omega_n > 2.705543$.

Tables 1 and 2 show the proportion of times that the AIC and BIC of Poisson-Weibull regression model was less than the exponential-Poisson and Weibull regression models. Furthermore, it was calculated the power of the test with 5% of the significance for different samples sizes, percentages of the censored observations and different parameter values of the new regression model. Through the Tables, it was noted that the AIC proportion of the Poisson-Weibull regression model was lower than the exponential-Poisson regression model in values between 35.01% and 99.98%. Moreover, the higher proportions of the AIC was in the scenario in which the parameters were $\alpha = 5$ and $\sigma = 0.80$, with the minimum proportion of the 58.36%. On the other hand, lower values of the AIC occurred in the situation that the size of n was smaller than 130, independent of the scenarios. However, in relation to BIC, the proportion varied between 0.1% and 71.76%. This variability was due to the change the value of σ (0.80 to 0.85). In situation that σ was equal to 0.85, the proportion of the BIC reached a maximum of the 25.08%. For $n > 200$ the proportion increased around of the 5%, in addition, the highest BIC proportions were in the scenario in which the parameters were $\alpha = 5$ and $\sigma = 0.80$, in particular, when n was increased. It was noted as well that the AIC selection criteria was better than the BIC for Poisson-Weibull regression model independent of the size of n and % censored observations.

In relation to the power of the test, the values varied between 14.55% and 98.81%. However, in the situation that σ was 0.80 and $\alpha = 5.0$, the power of the minimum test was 32.25%. In most cases, the power was greater in situations where σ was 0.80. Lower values of AIC, BIC and the power of the test occurred in the situation that the size of n was smaller than 130, independent of the scenarios.

Analyzing the Poisson-Weibull and Weibull regression models, it was noted that the AIC proportion of the Poisson-Weibull regression model was lower than the Weibull regression model in values between 50.32% and 95.22%. Furthermore, the values of the proportions were similar regardless of the σ.

But in relation to BIC, the proportion varied between 12.06% and 69.21%. For $n > 200$, the proportion increased around 45%, reducing its variability. The highest proportions of BIC occurred when the parameters were $\alpha = 5$ and $\sigma = 0.80$, in particular when n assumed values 400 and 500. We also noted that the AIC selection criteria was better than the BIC regardless of the size of n and % censored observations. In relation to the power of the test, regardless of the values of the parameters, the minimum value was 51.11%, which was a good power. Smaller proportions of AIC, BIC and the power of the test occurred in the situation that the size of the n was smaller than 130. For all sample sizes, the convergence rate was calculated for all scenarios, and it was found that the rate was 100% in all cases.

Also in this paper, a simulation study was done to calculate the size of the test for different values of n with 0%, 10% and 30% censored observations in each sample generating 10000 random samples simulated of Poisson-Weibull regression model, compared with exponential-Poisson and Weibull regression models. In relation to the Poisson-Weibull and exponential Poisson regression models, the test sizes were close to 5%, especially in situations that we increased the sample size and/or the decreased % of censored observations. However, for the Poisson-Weibull and Weibull regression models the test sizes were around

5% only in situations that the observations were not censored, because when we included censored observations, the test sizes had increased considerably.

Table 1. Simulated AIC, BIC and size and powers of the likelihood-ratio test for Poisson-Weibull, exponential-Poisson and Weibull regression models with $\sigma = 0.80$, $\alpha = 3.5$ and $\alpha = 5$.

				exponential-Poisson model			**Weibull model**		
σ	α	n	% of censorship	AIC	BIC	Power of the test	AIC	BIC	Power of the test
0.80	3.5	60	0%	0.5931	0.0280	0.3279	0.6919	0.3190	0.5111
			10%	0.5520	0.0271	0.2989	0.6800	0.2343	0.6423
			30%	0.5357	0.2006	0.2875	0.5032	0.3866	0.5899
		130	0%	0.7981	0.0551	0.5981	0.5569	0.4831	0.7232
			10%	0.7832	0.0349	0.5705	0.7645	0.4302	0.7398
			30%	0.7823	0.0381	0.5771	0.6248	0.4769	0.6144
		200	0%	0.9172	0.1231	0.7935	0.8438	0.0621	0.7830
			10%	0.8918	0.1111	0.7489	0.7872	0.5288	0.7715
			30%	0.8650	0.0982	0.7125	0.6319	0.5242	0.6227
		300	0%	0.9789	0.2999	0.8800	0.9210	0.4701	0.8748
			10%	0.9597	0.2010	0.8597	0.7991	0.5340	0.7842
			30%	0.9261	0.2001	0.8611	0.5889	0.5261	0.5843
		400	0%	0.9905	0.4169	0.9333	0.9516	0.6466	0.9304
			10%	0.9722	0.3998	0.9171	0.8085	0.6619	0.8004
			30%	0.9349	0.3061	0.9099	0.6441	0.5827	0.6404
		500	0%	0.9862	0.5695	0.9722	0.9716	0.6921	0.9594
			10%	0.9680	0.5641	0.9822	0.8102	0.6867	0.8025
			30%	0.9388	0.4908	0.9668	0.6348	0.5860	0.6348
	5.0	60	0%	0.6203	0.0285	0.3899	0.6356	0.3015	0.5645
			10%	0.6031	0.0232	0.3656	0.6481	0.2230	0.6043
			30%	0.5836	0.0111	0.3225	0.5960	0.3999	0.5745
		130	0%	0.8401	0.0998	0.6359	0.8401	0.1153	0.7862
			10%	0.8229	0.0579	0.6313	0.7242	0.4110	0.6985
			30%	0.7931	0.0628	0.5538	0.5834	0.4503	0.5738
		200	0%	0.9396	0.1956	0.8039	0.8978	0.3010	0.8704
			10%	0.9069	0.1631	0.7801	0.7491	0.5121	0.7343
			30%	0.8980	0.2767	0.7801	0.5915	0.4960	0.5853
		300	0%	0.9789	0.3771	0.9162	0.8932	0.5932	0.8767
			10%	0.9691	0.3630	0.8883	0.7571	0.5543	0.7894
			30%	0.9182	0.2868	0.8773	0.5762	0.5235	0.5759
		400	0%	0.9821	0.5811	0.9713	0.9341	0.5881	0.9099
			10%	0.9719	0.5780	0.9660	0.7901	0.6342	0.7655
			30%	0.9658	0.6324	0.9856	0.5583	0.5240	0.5548
		500	0%	0.9982	0.7176	0.9881	0.9522	0.5398	0.9109
			10%	0.9901	0.7001	0.9866	0.7851	0.5058	0.9069
			30%	0.9781	0.6734	0.9661	0.7991	0.5001	0.8899

Additionally, the mean, relative bias and MSE (Mean Square Error) of the maximum likelihood estimative were also calculated for simulated samples in the same conditions of the previous simulations. A simulation study was done for 60, 130, 200 and 300 with 0%, 10% and 30% censored observations in each sample. The censure time C is a random variable exponential with parameter $\lambda = 4$. The parameter values were fixed in $\alpha = 3$, $\sigma = 0.3$ and 0.75, with $\mu_i = \beta_0 + \beta_1 x_i$, where x_i is the uniform distribution on the interval $(0, 1)$ with β_0 and β_1 fixed in 0.8 and 1.5, respectively. For each combination of parameter values with the censoring percentages, 10000 samples were generated and were obtained the maximum likelihood estimates of Poisson-Weibull regression model.

From the simulation results, shown in Tables 3 and 4, it was observed that the estimates of the parameters of Poisson-Weibull regression model were close to the true value of the parameters. In addition, the MSE increased as the censoring percentages increased. On the other hand, the MSE decreased as the values of n increased. It was also observed that in most cases the estimates and the MSE values of the model parameters were closer of the

Table 2. Simulated AIC, BIC and size and powers of the likelihood-ratio test for Poisson-Weibull, exponential-Poisson and Weibull regression models with $\sigma = 0.85$, $\alpha = 3.5$ and $\alpha = 5$.

				exponential-Poisson model			**Weibull model**		
σ	α	n	% of censorship	AIC	BIC	Power of the test	AIC	BIC	Power of the test
			0%	0.3602	0.0051	0.1575	0.6532	0.1242	0.5843
		60	10%	0.3501	0.0010	0.1455	0.6520	0.2730	0.6082
			30%	0.3695	0.0073	0.1685	0.5740	0.3409	0.5569
			0%	0.5493	0.1002	0.3079	0.7756	0.0801	0.7129
		130	10%	0.5382	0.0093	0.2829	0.7433	0.4159	0.7186
			30%	0.5071	0.0072	0.2432	0.6050	0.4604	0.5961
			0%	0.6871	0.0128	0.4409	0.8251	0.1206	0.7615
		200	10%	0.6693	0.0110	0.4224	0.7673	0.5267	0.7508
			30%	0.6326	0.0051	0.3699	0.6076	0.5066	0.6004
			0%	0.7953	0.0429	0.5867	0.9176	0.2426	0.8672
		300	10%	0.8139	0.0314	0.5897	0.7785	0.6159	0.7726
	3.5		30%	0.7514	0.0253	0.5255	0.5994	0.5339	0.5939
			0%	0.8932	0.0772	0.7533	0.9529	0.2470	0.9300
		400	10%	0.8806	0.0815	0.7151	0.7919	0.6434	0.7836
			30%	0.8443	0.0661	0.6494	0.6305	0.5721	0.6266
			0%	0.9330	0.1340	0.8000	0.9596	0.5081	0.9433
		500	10%	0.9236	0.1239	0.8098	0.8092	0.6819	0.8114
			30%	0.9209	0.0891	0.7516	0.6165	0.5711	0.6138
			0%	0.3875	0.0109	0.1818	0.6212	0.1982	0.5476
0.85		60	10%	0.3896	0.0088	0.1813	0.6290	0.2500	0.5760
			30%	0.4011	0.0037	0.1862	0.5108	0.3387	0.5244
			0%	0.5949	0.0219	0.3260	0.7153	0.1662	0.6799
		130	10%	0.5861	0.0138	0.3305	0.7003	0.4198	0.5679
			30%	0.5687	0.0160	0.3244	0.5646	0.4662	0.6177
			0%	0.7345	0.0351	0.4901	0.7912	0.1780	0.7730
	5.0	200	10%	0.7313	0.0242	0.4773	0.7482	0.5198	0.7001
			30%	0.6942	0.0018	0.4622	0.7222	0.4843	0.5578
			0%	0.8624	0.0777	0.6794	0.8093	0.2822	0.5828
		300	10%	0.8077	0.0647	0.6694	0.7113	0.5697	0.7032
			30%	0.7617	0.0554	0.5864	0.5433	0.4778	0.5403
			0%	0.9053	0.1881	0.8022	0.9315	0.3019	0.9064
		400	10%	0.8921	0.1414	0.7137	0.7587	0.6328	0.7518
			30%	0.8909	0.0925	0.6228	0.5819	0.5317	0.5792
			0%	0.9278	0.2508	0.8268	0.8479	0.5519	0.8009
		500	10%	0.9125	0.2175	0.8021	0.7521	0.5563	0.7439
			30%	0.9078	0.1939	0.7909	0.5679	0.5581	0.5466

true value for the $\sigma = 0.3$, in comparison with $\sigma = 0.75$. Higher values of MSE occurred in the situation that the size of n was smaller than 130. In relation to the relative bias, their values remained close when was compared to the $\sigma = 0.3$, with $\sigma = 0.75$ considering all scenarios. The greatest impact of the relative bias occurred with the parameter α, because their values were higher in all combinations. In contrast, the lowest occurred with the parameter β_1. Also, higher values of the relative bias occurred in the situation that the size of n was smaller than 130 independent of the combinations.

7. Sensitivity analysis

After fitting the model, it is important to check its assumptions and to conduct a robustness study to detect influential or extreme observations that can cause distortions to the results of the analysis. The first tool to perform sensitivity analysis is the global influence starting from case deletion, see, Cook (1977). Case deletion is a common approach to study the effect of dropping the ith case from the data set. Another approach was suggested by Cook (1986), where instead of removing observations, weights are given to them. This is a local influence approach. Local influence calculation can be carried out in the model (6).

Table 3. Mean, Relative bias and MSE (Mean Square Error) of the estimates of the parameters of Poisson-Weibull regression model with $\alpha = 3$, $\sigma = 0.3$, $\beta_0 = 0.8$ and $\beta_1 = 1.5$.

		0%			10%			30%		
n	Parameters	Mean	Bias(%)	$M.S.E$	Mean	Bias(%)	$M.S.E$	Mean	Bias(%)	$M.S.E$
60	α	2.4832	-17.2266	15.7788	2.3331	-22.2300	14.8052	2.3357	-22.1433	23.8606
	σ	0.3033	1.1000	0.0027	0.3031	1.0333	0.0026	0.3007	0.2333	0.0028
	β_0	0.7164	-10.4500	0.0470	0.7059	-11.7625	0.0477	0.6929	-13.3875	0.0549
	β_1	1.5016	0.1066	0.0297	1.4989	-0.0733	0.0323	1.4996	-0.0266	0.0410
130	α	2.6869	-10.4366	4.9587	2.6555	-11.4833	5.1674	2.5867	-15.2833	8.2129
	σ	0.3050	1.6666	0.0018	0.3047	1.5666	0.0017	0.3039	1.3666	0.0021
	β_0	0.7522	-5.9750	0.0321	0.7489	-6.3875	0.0325	0.7372	-8.0750	0.0383
	β_1	1.5001	0.0066	0.0130	1.4994	-0.0400	0.0143	1.5003	0.1933	0.0186
200	α	2.7916	-3.8666	3.3285	2.7768	-7.4400	3.7947	2.7152	-9.4933	5.0734
	σ	0.3049	1.8333	0.0008	0.3047	1.5666	0.0012	0.3045	1.5000	0.0013
	β_0	0.7666	-3.2625	0.0242	0.7644	-4.4500	0.0264	0.7558	-5.5250	0.0298
	β_1	1.5007	-0.1133	0.0085	1.5008	0.0533	0.0094	1.5002	0.0133	0.0117
300	α	2.9398	-2.0066	2.9598	2.9194	-2.6866	3.1422	2.8611	-4.6300	3.5927
	σ	0.3043	1.4333	0.0006	0.3047	1.5666	0.0006	0.3048	1.6000	0.0006
	β_0	0.7818	-2.2750	0.0202	0.7803	-2.4625	0.0215	0.7730	-3.3750	0.0235
	β_1	1.5012	0.0800	0.0056	1.5001	0.0066	0.0060	1.4994	-0.0400	0.0079

Table 4. Mean, Relative bias and MSE (Mean Square Error) of the estimates of the parameters of Poisson-Weibull regression model with $\alpha = 3$, $\sigma = 0.75$, β_0=0.8 and β_1=1.5.

		0%			10%			30%		
n	Parameters	Mean	Bias(%)	$M.S.E$	Mean	Bias(%)	$M.S.E$	Mean	Bias(%)	$M.S.E$
60	α	2.3694	-21.0200	9.9438	2.3324	-22.2533	12.8394	2.3307	-22.3100	43.0302
	σ	0.7590	1.2000	0.0127	0.7606	1.4133	0.0121	0.7573	0.9733	0.0145
	β_0	0.5805	-27.4375	0.2680	0.5655	-29.3125	0.3016	0.4985	-37.6875	0.4064
	β_1	1.4917	-0.5533	0.1892	1.4922	-0.5200	0.2029	1.4901	-0.6600	0.2428
130	α	2.7220	-9.2666	4.9356	2.6392	-12.0266	5.2657	2.4662	-17.7933	8.2289
	σ	0.7613	1.5066	0.0103	0.7663	2.1733	0.0113	0.7580	1.0666	0.0114
	β_0	0.6880	-14.0000	0.1917	0.6648	-16.8875	0.2121	0.6133	-23.3375	0.2383
	β_1	1.4968	-0.2133	0.0860	1.5034	0.2266	0.0900	1.4938	-0.4133	0.1144
200	α	2.7860	-7.1333	4.7776	2.7554	-8.1533	3.4372	2.6462	-11.7933	7.0821
	σ	0.7642	1.8933	0.0091	0.7646	1.9466	0.0060	0.7620	1.6000	0.0099
	β_0	0.7364	-7.9500	0.1576	0.7075	-11.5625	0.1659	0.6568	-17.9000	0.2069
	β_1	1.4992	-0.0533	0.0501	1.5008	0.0533	0.0058	1.5368	2.4533	0.0712
300	α	2.9730	-0.9000	2.8928	2.9501	-1.6633	2.9796	2.9001	-3.3300	3.9553
	σ	0.7520	0.2666	0.0040	0.7608	1.4400	0.0021	0.7604	1.3866	0.0068
	β_0	0.7556	-5.5500	0.1478	0.7250	-9.375	0.1678	0.6911	-13.6125	0.1684
	β_1	1.5004	0.0266	0.0360	1.5056	0.3733	0.0376	1.5120	0.80000	0.0871

Consider a perturbation vector $\omega = (\omega_1, \ldots, \omega_n)^T$ varying in an open region $\mathbf{\Omega} \subset R^n$. If likelihood displacement $LD(\boldsymbol{\omega}) = 2\{l(\hat{\boldsymbol{\theta}}) - l(\hat{\boldsymbol{\theta}}_{\boldsymbol{\omega}})\}$ is used, where $\hat{\boldsymbol{\theta}}_{\boldsymbol{\omega}}$ denotes the MLE under the perturbed model, the normal curvature for $\boldsymbol{\theta}$ at direction $\mathbf{d}$, $\| \mathbf{d} \| = 1$, is given by $C_{\mathbf{d}}(\boldsymbol{\theta}) = 2|\mathbf{d}^T \boldsymbol{\Delta}^T \left[\ddot{\mathbf{L}}(\boldsymbol{\theta})\right]^{-1} \boldsymbol{\Delta}\mathbf{d}|$. Here, $\boldsymbol{\Delta}$ is a $k \times n$ matrix that depends on the perturbation scheme, and whose elements are given by $\Delta_{ji} = \partial^2 l(\boldsymbol{\theta}|\boldsymbol{\omega})/\partial\theta_j \partial\omega_i$, for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, k$, evaluated at $\hat{\boldsymbol{\theta}}$ and $\boldsymbol{\omega}_0$, where $\boldsymbol{\omega}_0$ is the no perturbation vector.

We can calculate normal curvatures $C_{\mathbf{d}}(\boldsymbol{\theta})$, $C_{\mathbf{d}}(\boldsymbol{\beta})$, $C_{\mathbf{d}}(\sigma)$ and $C_{\mathbf{d}}(\alpha)$ to perform various index plots, for instance, the index plot of $\mathbf{d}_{max}$, the eigenvector corresponding to $C_{\mathbf{d}_{max}}$, the largest eigenvalue of the matrix $\mathbf{B} = -\boldsymbol{\Delta}^T \left[\ddot{\mathbf{L}}(\boldsymbol{\theta})\right]^{-1} \boldsymbol{\Delta}$ and the index plots of $C_{\mathbf{d}_i}$, called the total local influence (see, for example, Lesaffre and Verbeke (1998)), where $\mathbf{d}_i$ denotes an $n \times 1$ vector of zeros with one at the ith position. Thus, the curvature at direction $\mathbf{d}_i$ assumes the form $C_i = 2|\boldsymbol{\Delta}_i^T \left[\ddot{\mathbf{L}}(\boldsymbol{\theta})\right]^{-1} \boldsymbol{\Delta}_i|$, where $\boldsymbol{\Delta}_i^T$ denotes the ith row of $\boldsymbol{\Delta}$.

There is some inconvenience when using the normal curvature to decide about the influence of the observations, since $C_{\mathbf{d}}(\boldsymbol{\theta})$ may assume any value and it is not invariant under a uniform change of scale. Based on the work of Poon and Poon (1999), using the following conformal normal curvature:

$$B_{\mathbf{d}}(\boldsymbol{\theta}) = \frac{C_{\mathbf{d}}(\boldsymbol{\theta})}{trace\big[\big(\boldsymbol{\Delta}^T\big[\ddot{\mathbf{L}}(\boldsymbol{\theta})\big]^{-1}\boldsymbol{\Delta}\big)^T\big(\boldsymbol{\Delta}^T\big[\ddot{\mathbf{L}}(\boldsymbol{\theta})\big]^{-1}\boldsymbol{\Delta}\big)\big]},$$

evaluated at $\hat{\boldsymbol{\theta}}$ and $\boldsymbol{\omega}_0$, whose computation is quite simple and also has the property that $0 \leq\mid B_{\mathbf{d}}(\boldsymbol{\theta}) \mid\leq 1$. This influence measure possesses an one-to-one relationship with the normal curvature proposed by Cook (1986). So, we can also calculate conformal normal curvatures to perform various index plots, for instance, the index plot of $\mathbf{d}_{max}$, the eigenvector corresponding to $B_{\mathbf{d}_{max}}$, the largest eigenvalue of the matrix

$$\mathbf{B} = \frac{\boldsymbol{\Delta}^T\big[\ddot{\mathbf{L}}(\boldsymbol{\theta})\big]^{-1}\boldsymbol{\Delta}}{trace\big[\big(\boldsymbol{\Delta}^T\big[\ddot{\mathbf{L}}(\boldsymbol{\theta})\big]^{-1}\boldsymbol{\Delta}\big)^T\big(\boldsymbol{\Delta}^T\big[\ddot{\mathbf{L}}(\boldsymbol{\theta})\big]^{-1}\boldsymbol{\Delta}\big)\big]}.$$

The conformal normal curvature gives reference values that allow one to evaluate the magnitude of a certain curvature. In this case, the conformal normal curvature influence $B_{\mathbf{E}_i}$ of the basic perturbation vector $\mathbf{E}_i$ is given by $B_{\mathbf{E}_i} = \sum_{i=1}^n \hat{\mu}_j a_{ji}^2$, where μ_j denotes the absolute value of the jth normalized eigenvalue of the matrix $\mathbf{B}$, a_{ji} denotes the ith element of the normalized eigenvector corresponding to μ_j and $\hat{\mu}_j = \mu_j/\sqrt{(}\sum_{k=1}^n \mu_k^2)$. Poon and Poon (1999) suggest using $2b$ as cut point of the aggregate contribution of all the $B_{\mathbf{E}_i}$, where $b = trace(\mathbf{B})/[trace(\mathbf{B}^T\mathbf{B})]^{(1/2)}$. Then, those cases when $B_{\mathbf{E}_i} > 2b$ are considered like potentially influential. For more details about this technique, see, Poon and Poon (1999).

More generally, we can analyse the influence of basic perturbation vectors to all influential eigenvectors. We arrange the absolute values of the normalized eigenvalues by $\mu_{max} \geq \mu_1 \geq \cdots \mu_k \geq q/\sqrt{(}n) > \mu_{k+1} \geq \cdots \mu_n$. The aggregate contribution of the jth basic perturbation vector to all q influential eigenvectors is $m(q)_j = \sqrt{(}\sum_{i=1}^k \mu_i a_{ij}^2)$. So, in those cases when $m(q) > \overline{m}(q)_j = \sqrt{(}\frac{1}{n}\sum_{i=1}^k \mu_i)$ are considered like potentially influential. For more details about this technique, see Poon and Poon (1999).

Now, we evaluate the following perturbation schemes: case-weight perturbation to detect observations with outstanding contribution of the log-likelihood function and that can exercise high influence on the maximum likelihood estimates; response perturbation of the response values, which can indicate observations with large influence on their own predicted values; and finally explanatory variables perturbation. The matrix $\boldsymbol{\Delta} = (\boldsymbol{\Delta}_\alpha, \boldsymbol{\Delta}_\sigma, \boldsymbol{\Delta}_{\boldsymbol{\beta}})^T$ can be obtained numerically.

Case-weight perturbation

In this case, for the Poisson-Weibull regression model, the log-likelihood function takes the form $l(\boldsymbol{\theta}|\boldsymbol{\omega}) = \sum_{i=1}^n \omega_i\, l(\boldsymbol{\theta})$, where $0 \leq \omega_i \leq 1$, $\boldsymbol{\omega}_0 = (1,\ldots,1)^T$. The elements of matrix $\boldsymbol{\Delta} = (\boldsymbol{\Delta}_\alpha, \boldsymbol{\Delta}_\sigma, \boldsymbol{\Delta}_{\boldsymbol{\beta}})^T$ are given in Appendix A.

Response perturbation

For the Poisson-Weibull regression model, each y_i is perturbed as $y_{iw} = y_i + \omega_i S_y$, where S_y is a scale factor that can be estimated by the standard deviation of Y and $\omega_i \in \mathbf{R}$. Here, $\boldsymbol{\omega}_0 = (0,\ldots,0)^T$. The elements of matrix $\boldsymbol{\Delta} = (\boldsymbol{\Delta}_\alpha, \boldsymbol{\Delta}_\sigma, \boldsymbol{\Delta}_{\boldsymbol{\beta}})^T$ are given in Appendix B.

Explanatory variable perturbation

Consider now an additive perturbation on a particular continuous explanatory variable, say X_t, by setting $x_{it\omega} = x_{it} + \omega_i S_t$, where S_t is a scaled factor, $\omega_i \in \mathbf{R}$. Here, $\boldsymbol{\omega}_0 = (0, \dots, 0)^T$. The elements of matrix $\boldsymbol{\Delta} = (\boldsymbol{\Delta}_\alpha, \boldsymbol{\Delta}_\sigma, \boldsymbol{\Delta}_{\boldsymbol{\beta}})^T$ are given in Appendix C.

8. Residual Analisys

An important step after the model formulation is the analysis of residual. It is used to see if there are differences in the assumptions made in the model proposed. In this work we consider the residual Cox-Snell. This residual is used to check the overall fit of the model (Cox and Snell, 1968). It is described as

$$r_i = -\log\left[S\left(y_i|x_i\right)\right], \qquad i = 1, \dots, n,$$

where $S(\cdot)$ is the survival function obtained by adjusted model. The residual Cox-Snell to Poisson-Weibull regression model is defined by

$$r_i = -\log\left(\frac{\exp\{\alpha \exp[-\exp((y - \mathbf{x}_i^T\boldsymbol{\beta})/\sigma)]\} - 1}{\exp(\alpha) - 1}\right), \qquad i = 1, \dots, n.$$

This residuals follow a standard exponential distribution if the fitted model is adequate Lawless (2003).

9. Applications

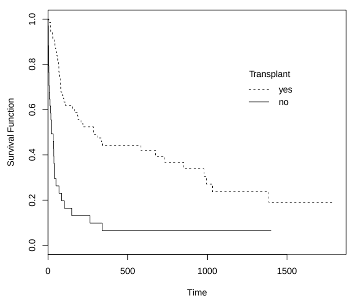
We illustrate the proposed model using data from Crowley and Hu (1977). The data referred to the survival time (in days) of the patients who were admitted into the Stanford University heart transplant program from 1967 to 1973. The aim of the study was to relate the survival time (t) for x_1: Year of acceptance to the program; x_2: Age of patients (in years); x_3: Previous Surgery (1=yes, 0=no) and x_4: If the patient did transplant in the program (1=yes, 0=no). The data contain $n = 103$ observations of which 27% are censored.

Initially, to get more information of the survival time was made an analysis these times, without considering the observations censored. This information is shown in Table 5. It can be observed in accordance with this table that the median time of patients was 66 days, which indicates that approximately 50% of patients had the survival time larger than 66 days, or approximately 50% of patients had the survival time at most 66 days and its mean survival time was 171.2 days. It can also be observed that 25% of patients had a lifetime less than 19.5 days, or greater than 175.5 days. Furthermore, the lifetime of the patients were between 1 and 1386 days, which implies a great variability over time. What certifies this great variability is the standard deviation value of 280.9684 days.

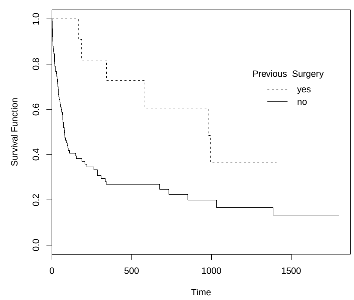
Figure 3 shows the survival estimates by Kaplan-Meier for groups of patients on the variables: Transplant in the program; Previous Surgery; and Age of patients. From this Figure, it was noted that there is evidence of difference between survival functions for groups of patients, for example, patients with previous surgery or no surgery. Therefore, these variables seem to influence the survival time.

Table 5. Summary of survival time of patients admitted into the Stanford University heart transplant program from 1967 to 1973.

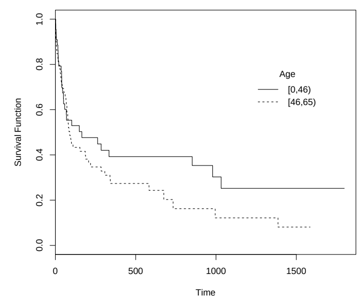
Minimum	First quartile	Median	Mean	Third quartile	Maximum	Standard deviation
1.0000	19.5000	66.0000	171.2000	175.5000	1386.0000	280.9684



(a) Transplant



(b) Previous Surgery



(c) Age of patients

Figure 3. Curve of the survival function estimated by the Kaplan-Meier to variables Transplant in the program, Previous Surgery and Age of patients.

An graphical analysis also was done using the TTT curve to verify the shape of the hazard rate function. According to Aarset (1987), the empirical version of the TTT plot is given by $G(r/n) = [(\sum_{i=1}^{r} Y_{i:n}) - (n-r)Y_{r:n}]/(\sum_{i=1}^{r} Y_{i:n})$, where $r = 1, \ldots, n$ and $Y_{i:n}$ represents the order statistics of the sample. Aarset (1987) showed that the hazard function is constant if the TTT plot is presented graphically as a straight diagonal; the hazard function is increasing (or decreasing) if the TTT plot is concave (or convex); the hazard function is U-shaped if the TTT plot is convex and then concave, otherwise, the hazard function is unimodal. The TTT plot for heart transplant data in Figure 4 indicates an decreasing shaped failure rate function. So, we can try using the Poisson-Weibull distribution for the modeling of data.

The next step after the conclusions made about the Poisson-Weibull regression model was to check which is the best model when covariates were included.

In this study, the model is expressed in the form:

$$y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \beta_3 x_{i3} + \beta_4 x_{i4} + \sigma z_i, \qquad i = 1, \ldots, 103,$$

where $y_i = \log(t_i)$ denotes the logarithm of the survival time.

We compared the Poisson-Weibull regression model with its particular cases exponential-Poisson and Weibull, considering the AIC. The preferred model is the one with the lowest AIC criterion. The values of AIC are presented in Table 6 with the maximum likelihood estimates and their standard errors for the parameters of the three models.

It can be observed in accordance with Table 6 that the Poisson-Weibull regression model had the lowest value of the AIC in relation the exponential-Poisson and Weibull models,

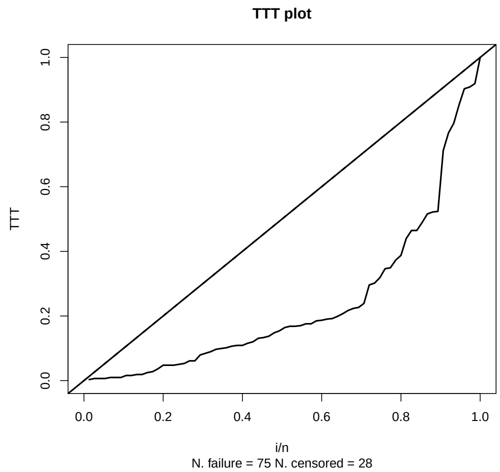


Figure 4. TTT plot to heart transplant data.

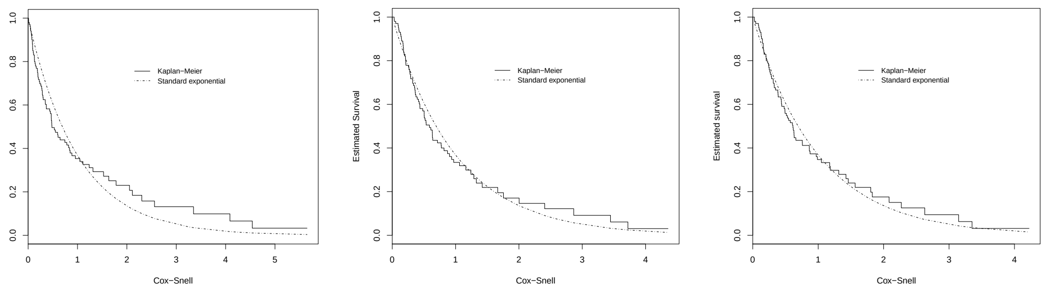
indicating that this model is more appropriate to the data. For exponential-Poisson and log Weibull regression models, the variables X_2 and X_4 were significant. For Poisson-Weibull regression model, the variables X_2, X_3 and X_4 were significant. The estimates of the parameters and their standard errors of the models were similar in most cases. For all models was used at 5% of significance.

Table 6. Estimated values of model parameters log exponential-Poisson, log Weibull and log Poisson-Weibull to data of heart transplant. (E.P. = standard errors)

	log exponential-Poisson			**log Weibull**			**log Poisson-Weibull**		
Parameters	Estimates	E.P.	p-value	Estimates	E.P.	p-value	Estimates	E.P.	p-value
α	3.2359	1.0821	-	-	-	-	3.0374	1.8982	-
σ	-	-	-	1.4760	0.1321	-	1.2813	0.1163	-
β_0	3.1733	5.5028	0.5641	1.3020	6.7664	0.8474	1.3105	6.9977	0.8514
β_1	0.0756	0.0779	0.3321	0.0945	0.0949	0.3194	0.1046	0.0988	0.2897
β_2	-0.0816	0.0165	< 0.0001	-0.0918	0.0203	< 0.0001	-0.0816	0.0205	< 0.0001
β_3	1.1425	0.4924	0.0203	1.1341	0.6573	0.0844	1.2479	0.6269	0.0465
β_4	2.4348	0.3085	< 0.0001	2.5422	0.3758	< 0.0001	2.5384	0.3885	< 0.0001
$-\ell(\cdot)$	173.3177			171.2324			169.0988		
AIC	358.6354			354.4648			352.1976		

As the Poisson-Weibull regression model is reduced in the regression models exponential-Poisson and Weibull, the likelihood ratio test was used to select models which best fit the data. This procedure is given in the Section 5. To the exponential-Poisson and Poisson Weibull regression models, hypotheses were: $H_0 : \sigma = 1$, i.e, the exponential-Poisson regression model is adequate versus $H_1 : \sigma \neq 1$, i.e, the Poison-Weibull regression model is adequate. In this data it was observed that the test statistic was 6.5539 (p-value = 0.0036) and this result leads us to reject the null hypothesis. In relation the Weibull and Poisson-Weibull regression models, hypotheses were: $H_0 : \alpha \to 0$, i.e, the Weibull regression model is adequate versus $H_1 : \alpha > 0$, i.e, the Poison-Weibull regression model is adequate. In this data it was observed that the test statistic was 4.2672 bigger than $1/2 + 1/2$ $P(\chi^2_1 \leq c) = 2.705543$, at the significance level of the 5%, which leads us to reject the null hypothesis. In both cases there is evidence in favor of the Poisson-Weibull regression model.

The next step after the conclusions made anteriorly about the models was the residual analysis, which is useful to verify the goodness of fit of the model. Figure 5 shows the Cox-Snell residual plot for the exponential-Poisson, Weibull and Poisson-Weibull models, respectively. From this Figure, its was noted that for the model Poisson-Weibull, the exponential curve is closer to the estimated survival curve in relation the exponential-Poisson and Weibull models.



(a) exponential-Poisson regression model

(b) Weibull regression model

(c) Poisson-Weibull regression model

Figure 5. Curve of the survival function of the residual estimated by the Kaplan-Meier estimated survival curve and the exponential standard of the exponential-Poisson, Weibull and Poisson-Weibull models to data of heart transplant.

Therefore, when removed the non-significant covariate, the estimates of the parameters of the Poisson-Weibull regression model to data of heart transplant are presented in Table 7.

Table 7. Estimated values of model parameters final of the Poisson-Weibull regression model to data of heart transplant.

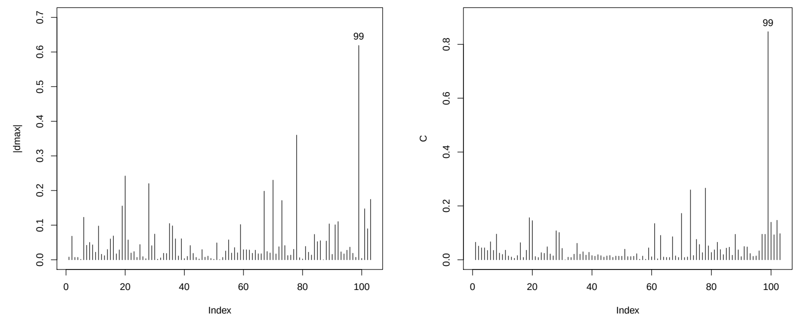
Parameters	Estimates	E.P.	p-value
α	3.0547	1.8524	-
σ	1.2744	0.1157	-
β_0	8.6522	1.0497	< 0.0001
β_1	-0.0818	0.0205	< 0.0001
β_2	1.3268	0.61642	0.0313
β_3	2.5734	0.3836	< 0.0001

From the considerations mentioned, the final model is described by:

$$\hat{y}_i = \log(t_i) = 8.6522 - 0.0818x_{i1} + 1.3268x_{i2} + 2.5734x_{i3}, \qquad i = 1, \ldots, 103.$$

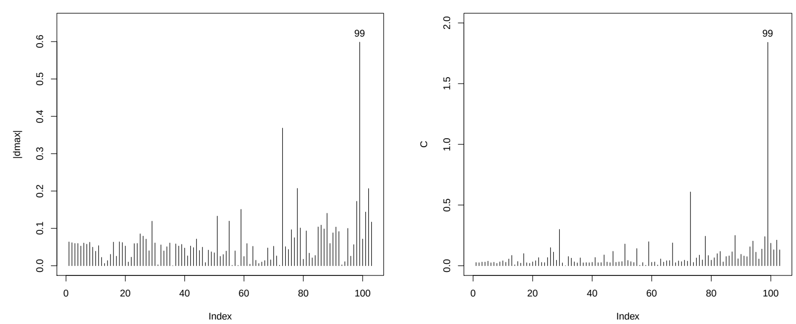
Next, we conduct a local influence study as described in Section 7 by considering the three different perturbation schemes. We used the software Ox to compute the measures presented in this section. In Figure 6a, the graph of eigenvector corresponding to $C_{\mathbf{d}_{max}}$ is presented, and total influence C_i is shown in figure 6b. The observation 99 was the most distinguished in relation to the others.

Subsequently, we analyzed the influence of perturbations on the observed survival times. Figure 7(a) contains the graph for $C_{\mathbf{d}_{max}}$ versus the observation index, showing that the observation 99 was the most salient in relation to the others. Figure 7(b) presents graphs for the total local influence (Ci), where the observation 99 again stands out. The perturbation of vectors for continuous explanatory variable age was investigated here. The respectives graphs of $\mathbf{d}_{max}$ as well as total local influence C_i against the observation index are shown in Figures 8(a) and 8(b). Note that observations 29; 73 and 99 clearly stands out for the explanatory variable age.



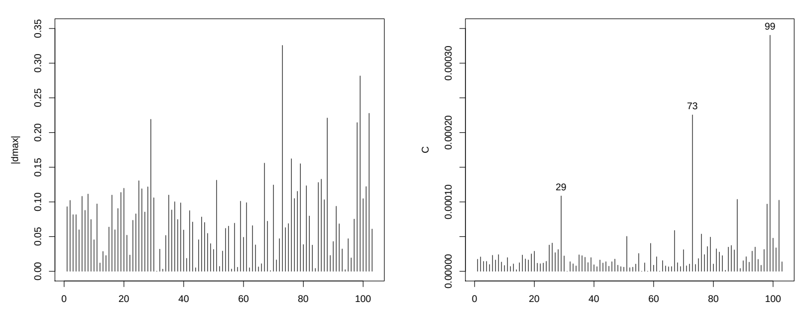
(a) Local influence (b) Total local influence

Figure 6. Index plot for $\boldsymbol{\theta}$ (case-weight perturbation): Local Influence and Total local influence.



(a) Local influence (b) Total local influence

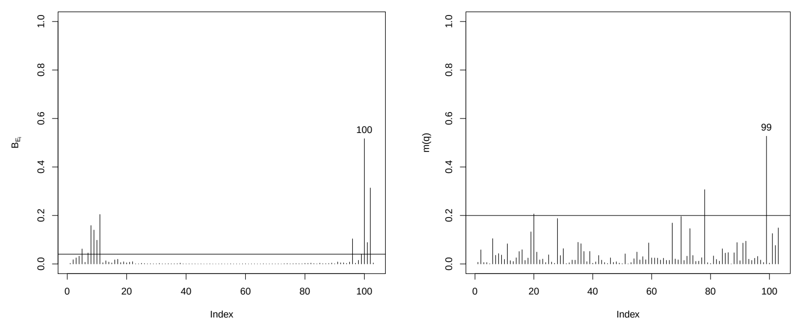
Figure 7. Index plot for $\boldsymbol{\theta}$ (response perturbation): Local Influence and Total local influence.



(a) Local influence (b) Total local influence

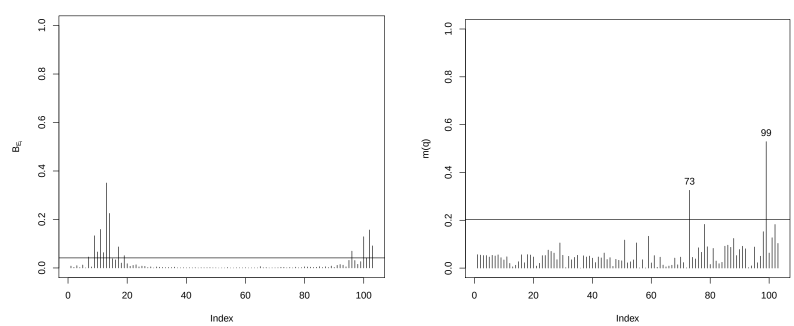
Figure 8. Index plot for $\boldsymbol{\theta}$ (age explanatory variable perturbation):Local Influence and Total local influence.

For each case, the conformal normal curvature of the basic perturbation vector B_E and the aggregate contributions $m(q)_j$, with $q = 1$ and $q = 7$, were computed. When $q = 7$, only the largest eigenvalue is considered as influential and when $q = 1$, it is choosen only non-zero eigenvalues. Figures 9(a) - 11(a) present values for the added contribution of all the eigenvalues B_{E_i}, for $i = 1, \ldots, 103$, against the index of the observations. Observations 7 and 100 are the most distinguished in relation to the others. However, in Figures 9(b) - 11(b), where the index plots of $m_q(j)$ for $q = 1$ under the all perturbation schemes are displayed, only observation 99 appear as potentially influential. Figures corresponding to $q = 7$ have been omitted in this work because the results are similar to those in that $q = 1$. Note that, there are many observations bigger than cut point, so it was choosed only the points than most distinguished in relation to the others.



(a) Conformal local influence (b) Conformal local influence q=1

Figure 9. Index plot for $\boldsymbol{\theta}$ (case-weight perturbation): Conformal local influence for all eigenvalues and Conformal local influence for q=1.



(a) Conformal local influence (b) Conformal local influence q=1

Figure 10. Index plot for $\boldsymbol{\theta}$ (response perturbation): Conformal local influence for all eigenvalues and Conformal local influence for q=1.

Upon conclusion of previous results, we can consider the cases 7; 29, 73, 99 and 100 as possible influential or outlier observations. These observations were identified as possible influential points. Thus, are patients that have the following characteristics:

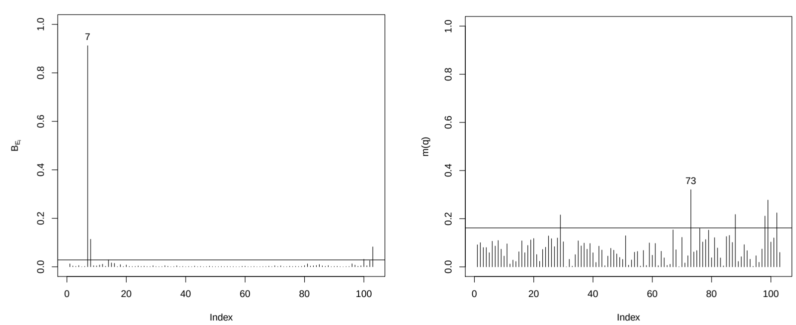
(i) The observation 7 matches to the patient that did not have censure, with age of 50 years, did not surgery before and did the transplant. This observation has a lower survival time of the patients with these caracteristics.

(ii) The observation 29 matches to the patient that did not have censure, with age of 59 years, did not surgery before and did not the transplant. This observation is the oldest of the patients with these caracteristics.

(iii) The observation 73 matches to the patient that did not have censure, with age of 47 years, did not surgery before and did not the transplant. This observation has the largest survival time of the patients with these caracteristics.

(iv) The observation 99 matches to the patient that did have censure, with age of 30 years, did not surgery before and did not the transplant. This observation has the largest survival time of the patients with these caracteristics.

(v) The observation 100 matches to the patient that do not have censure, with age of 30 years, did surgery before and did the transplant. This observation has the largest survival time of the patients with these caracteristics.



(a) Conformal local influence (b) Conformal local influence q=1

Figure 11. Index plot for $\boldsymbol{\theta}$ (age explanatory variable perturbation): Conformal local influence for all eigenvalues and Conformal local influence for q=1.

To verify if these observations are possible influential points in order to reveal the impact of these observations on the parameter estimates, several combinations of candidates exclusions were made, and the parameters of model were again estimated when it was eliminated one, two, until five observations. The Tables 8 and 9 show the maximum likelihood estimates and their p-values (among parentheses) these combinations. It can be observed in accordance with this table that when the observations 7, 29, 73 and 99 were removed individually of the data set, the estimates remained close in relation to the original data. Also, the significance of the parameters did not change compared to the complete data. For the observation 100, the estimates also remained near when compared with the original data, not changing the conclusions about the significance of most parameters, with the exception of the parameter β_2. The same results were obtained when it was eliminated the observation 100 along with other observations. For example, when are removed together the observations: (7;100); (29;100); (73;100); (99;100); (7;73;100); (29;99;100); (73;99;100); (7;29;73;100); (7;73;99;100); (7;29;99;100); (29;73;99;100) and (7;29;73;99;100), the parameter β_2 is not significant.

The Table 9 shows the percentage change of each estimated parameter, that is given by $\left[\left(\widehat{\theta}_j - \widehat{\theta}_{j(i)}\right)/\widehat{\theta}_j\right] X100$, in which $\widehat{\theta}_j$ is the estimate of the maximum likelihood with all observations, and $\widehat{\theta}_{j(i)}$ is the estimate of the maximum likelihood without the i_{th} ob-

Table 8. Values of the maximum likelihood estimates and p-values of parameters of the Poisson-Weibull regression model.

	Parameters					
Data	α	σ	β_0	β_1	β_2	β_3
A=Complete	3.0547	1.2744	8.6522	-0.0818	1.3268	2.5734
			(< 0.0001)	(< 0.0001)	(0.0313)	(< 0.0001)
A-{7}	3.0308	1.2678	8.6485	-0.0809	1.3251	2.5168
			(< 0.0001)	(< 0.0001)	(0.0307)	(< 0.0001)
A-{29}	3.1100	1.2774	8.8388	-0.0871	1.3216	2.6641
			(< 0.0001)	(< 0.0001)	(0.0324)	(< 0.0001)
A-{73}	3.3010	1.2475	8.6591	-0.0835	1.3094	2.7042
			(< 0.0001)	(< 0.0001)	(0.0299)	(< 0.0001)
A-{99}	2.6329	1.2124	7.6968	-0.0687	1.3147	2.6748
			(< 0.0001)	(< 0.0001)	(0.0255)	(< 0.0001)
A-{100}	3.0812	1.2682	8.6696	-0.0821	1.0995	2.5733
			(< 0.0001)	(< 0.0001)	(0.0739)	(< 0.0001)
A-{7; 29}	3.0805	1.2716	8.8257	-0.0860	1.3207	2.6054
			(< 0.0001)	(< 0.0001)	(0.0318)	(< 0.0001)
A-{7; 73}	3.2976	1.2420	8.6612	-0.0826	1.3081	2.6494
			(< 0.0001)	(< 0.0001)	(0.0293)	(< 0.0001)
A-{7; 99}	2.6021	1.2062	7.6945	-0.0680	1.3120	2.6222
			(< 0.0001)	(0.0003)	(0.0251)	(< 0.0001)
A-{7; 100}	3.0566	1.2615	8.6658	-0.0812	1.0986	2.5168
			(< 0.0001)	(< 0.0001)	(0.0728)	(< 0.0001)
A-{29; 73}	3.3368	1.2465	8.85765	-0.0896	1.3015	2.8086
			(< 0.0001)	(< 0.0001)	(0.0308)	(< 0.0001)
A-{29; 99}	2.7725	1.2158	7.8952	-0.0735	1.3117	2.7571
			(< 0.0001)	(0.0002)	(0.0262)	(< 0.0001)
A-{29; 100}	3.1390	1.2711	8.8571	-0.0874	1.0928	2.6646
			(< 0.0001)	(< 0.0001)	(0.0764)	(< 0.0001)
A-{73; 99}	1.9909	1.1736	7.3871	-0.0716	1.2828	2.8414
			(< 0.0001)	(< 0.0001)	(0.0251)	(< 0.0001)
A-{73; 100}	3.3376	1.2412	8.6783	-0.0838	1.0835	2.7043
			(< 0.0001)	(< 0.0001)	(0.0718)	(< 0.0001)
A-{99; 100}	2.6575	1.2064	7.7152	-0.0691	1.0950	2.6740
			(< 0.0001)	(0.0002)	(0.0626)	(< 0.0001)

servation. It can be observed in accordance with Table 10 that, individually, the lowest impact of the percentage change occurred when the observation 7 was removed, or when the observations that have this individual were eliminated. In contrast, the greatest impact individually occurred when the observation 99 was removed or when the observations that have this individual were eliminated. To variables in study, the variable Previous Surgery, in most cases, had the greatest impact when the observations were removed together, underestimating the parameter β_2.

Thus, from the analysis made, we considered the observation 100 as an influence point and removed of the dataset. Then, the parameters of model were again estimated without the variable Previous Surgery, as shown in Table 11. It can be observed in accordance with this Table that the variables Age of patients and Transplant in the program were

Table 9. Values of the maximum likelihood estimates and p-values of parameters of the Poisson-Weibull regression model.

	Parameters					
Data	α	σ	β_0	β_1	β_2	β_3
A=Complete	3.0547	1.2744	8.6522	-0.0818	1.3268	2.5734
			(< 0.0001)	(< 0.0001)	(0.0313)	(< 0.0001)
A-$\{7; 29; 73\}$	3.3271	1.2420	8.8523	-0.0885	1.3001	2.7525
			(< 0.0001)	(< 0.0001)	(0.0304)	(< 0.0001)
A-$\{7; 29; 99\}$	2.7333	1.2102	7.8835	-0.0725	1.3102	2.7025
			(< 0.0001)	(0.0002)	(0.0257)	(< 0.0001)
A-$\{7; 29; 100\}$	3.1084	1.2652	8.8445	-0.0863	1.0926	2.6059
			(< 0.0001)	(< 0.0001)	(0.0752)	(< 0.0001)
A-$\{7; 73; 99\}$	1.9878	1.1685	7.3947	-0.0708	1.2811	2.7912
			(< 0.0001)	(0.0001)	(0.0248)	(< 0.0001)
A-$\{7; 73; 100\}$	2.6257	1.2000	7.7121	-0.0683	1.0933	2.6216
			(< 0.0001)	(0.0001)	(0.0752)	(< 0.0001)
A-$\{29; 73; 99\}$	2.0684	1.1709	7.5806	-0.0771	1.2785	2.9375
			(< 0.0001)	(< 0.0001)	(0.0253)	(< 0.0001)
A-$\{29; 73; 100\}$	3.3766	1.2401	8.8781	-0.0899	1.0744	2.8092
			(< 0.0001)	(< 0.0001)	(0.0253)	(< 0.0001)
A-$\{29; 99; 100\}$	2.8021	1.2096	7.9158	-0.0738	1.0908	2.7570
			(< 0.0001)	(0.0001)	(0.0641)	(< 0.0001)
A-$\{73; 99; 100\}$	1.9894	1.1671	7.3972	-0.0720	1.0629	2.8413
			(< 0.0001)	(< 0.0001)	(0.0618)	(< 0.0001)
A-$\{7; 29; 73; 99\}$	2.0681	1.1669	7.5812	-0.0761	1.2774	2.8853
			(< 0.0001)	(< 0.0001)	(0.0250)	(< 0.0001)
A-$\{7; 29; 73; 100\}$	3.3693	1.2355	8.8729	-0.0888	1.0744	2.7528
			(< 0.0001)	(< 0.0001)	(0.0730)	(< 0.0001)
A-$\{7; 73; 99; 100\}$	1.9805	1.1623	7.4013	-0.0711	1.0615	2.7910
			(< 0.0001)	(< 0.0001)	(0.0636)	(< 0.0001)
A-$\{7; 29; 99; 100\}$	2.7622	1.2039	7.9039	-0.0729	1.0898	2.7024
			(< 0.0001)	(0.0002)	(0.0632)	(< 0.0001)
A-$\{29; 73; 99; 100\}$	2.0676	1.1647	7.5890	-0.0774	1.0581	1.0581
			(< 0.0001)	(0.0002)	(0.0632)	(< 0.0001)
A-$\{7; 29; 73; 99; 100\}$	2.0656	1.1605	7.5904	-0.0765	1.0578	2.8856
			(< 0.0001)	(< 0.0001)	(0.0641)	(< 0.0001)

significant, using a significance level of 5%.

From the considerations mentioned, the final model is described by:

$$\hat{y}_i = \log(t_i) = 8.7702 - 0.0842x_{i1} + 2.7947x_{i2}, \qquad i = 1, \ldots, 102.$$

According to the final model, the interpretation of the variable age of patients without to consider other variables was that the time median estimated survival should decrease approximately 8.78% ([exp(0.0842) x 100%]) when we increased one year. The interpretation of the variable transplant in the program without to consider other variables was that the median estimated survival of patients who did transplant in the program was approximately $\exp(2.7947) = 16.3577$ times higher than those who did not do the transplant.

Table 10. Percentage change of the maximum likelihood estimates of parameters of the Poisson-Weibull regression model.

	Parameters					
Data	α	σ	β_0	β_1	β_2	β_3
A-{7}	0.7796	0.5190	0.0432	0.0432	1.1059	0.1314
A-{29}	-1.8099	-0.2315	-99.7606	-6.4584	0.3988	-3.5265
A-{73}	-8.0649	2.1111	-0.0793	-2.0984	1.3170	-5.084
A-{99}	13.8073	4.8648	11.0421	15.9528	0.9193	-3.9415
A-{100}	-0.8682	0.4870	-0.2009	-0.3921	17.1302	0.0010
A-{7; 29}	-0.8470	0.2201	-2.0054	-5.0978	0.4669	-1.2467
A-{7; 73}	-7.9521	2.5452	-0.1034	-0.9684	1.4162	-2.9535
A-{7; 99}	14.8162	5.3564	11.0685	16.8914	1.1213	-1.8974
A-{7; 100}	-0.0620	1.0116	-0.1574	0.7042	17.2002	2.1984
A-{29; 73}	-9.2345	2.1893	-2.3739	-9.5487	1.9112	-9.1429
A-{29; 99}	9.2372	4.6008	8.7490	10.1666	1.1400	-7.1414
A-{29; 100}	-2.7620	1.2744	-2.3683	-6.8721	17.6386	-3.5459
A-{73; 99}	34.8233	7.9150	14.6217	12.4409	3.3218	-10.4173
A-{73; 100}	-9.2623	2.6043	-0.3009	-2.4738	18.3395	-5.0867
A-{99; 100}	13.0028	5.3356	10.8298	15.5616	17.4718	-3.9112
A-{7; 29; 73}	-8.9195	2.5442	-2.3127	-8.2117	2.01383	-6.9611
A-{7; 29; 99}	10.5199	5.0378	8.8847	11.3343	1.2553	-5.0184
A-{7; 29; 100}	-1.7589	0.7244	-2.2223	-5.5286	17.6510	-1.2662
A-{7; 73; 99}	34.9245	8.3092	14.5338	14.5338	3.4443	-8.4636
A-{7; 73; 100}	-9.2284	3.0416	-0.3283	-1.3332	18.3864	-2.9594
A-{29; 73; 99}	14.0424	5.8413	10.8651	16.5046	17.6028	-1.8731
A-{29; 73; 100}	-10.5385	2.6931	-2.6103	-9.9446	19.0271	-9.16409
A-{29; 99; 100}	8.2670	5.0839	8.5115	9.7380	17.7882	-7.1368
A-{73; 99; 100}	34.8712	8.4181	4.5053	12.0299	19.8937	-10.4126
A-{7; 29; 73; 99}	32.2980	8.4356	12.3783	6.9199	3.7300	-12.1810
A-{7; 29; 73; 100}	-10.2983	3.0573	-2.5508	-8.6001	19.0271	-12.1217
A-{7; 73; 99; 100}	35.1649	8.7967	14.4581	13.0283	19.9957	-8.4569
A-{7; 29; 99; 100}	9.5742	5.5319	8.6488	10.8978	17.8626	-5.0159
A-{29; 73; 99; 100}	32.3121	8.6061	12.2878	5.3385	20.2520	-14.1498
A-{7; 29; 73; 99; 100}	32.3798	8.9388	12.2721	6.5142	20.2740	-12.1349

Table 11. Estimated values of model parameters final of the Poisson-Weibull regression model to data of heart transplant.

Parameters	Estimates	E.P.	p-value
α	3.0812	2.2427	-
σ	1.2901	0.1178	-
β_0	8.7702	1.1376	< 0.0001
β_1	-0.0842	0.0212	< 0.0001
β_2	2.7947	0.3769	< 0.0001

10. Concluding remarks

In this paper, the Poisson-Weibull regression model, in the form location and scale with the presence of censored data, was proposed as an alternative to model lifetime and presented as particular cases the regression models exponential-Poisson and Weibull. We used maximum likelihood method for estimation of parameter. Asymptotic tests were performed for the parameters based on the asymptotic distribution of the maximum likelihood estimators. Additionally, this article compared the performance of the proposed model based on mean squared error (MSE), AIC, BIC and the likelihood ratio test through a simulation study. These simulations suggest that the Poisson-Weibull model can be used for modeling data with unimodal failure rate function. However, through the simulation studys, it was observed that more favorable results were obtained with the proportions of AIC, BIC, test power, MSE and bias relative for sample sizes larger than 130. In this study, we discussed the applications of influence diagnostics in Poisson-Weibull regression model with censored data. We also presented some ways to perform residual analysis. It was verified that the Poisson-Weibull regression model presented good performance in some cases. The approach was applied to the data set, which clearly indicated the usefulness of the approach. Thus, it is expected that this model will be used in other datasets.

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Appendix A. Appendix A: Case-weight perturbation scheme

Here, we provide the elements by considering the case-weight perturbation scheme. The elements of matrix $\mathbf{\Delta} = (\mathbf{\Delta}_1^T, \mathbf{\Delta}_2^T, \mathbf{\Delta}_j^T)^T$ are expressed as

$$\begin{aligned}
\Delta_{1i} &= \frac{\delta_i}{\hat{\alpha}} + \delta_i \exp\{-\exp(\hat{z}_i)\} - \delta_i \frac{\exp(\hat{\alpha})}{(\exp(\hat{\alpha}) - 1)\sigma} - (1-\delta_i)\frac{\exp(\hat{\alpha})}{\exp(\hat{\alpha}) - 1} \\
&\quad - (1-\delta_i)\frac{\exp[\hat{\alpha}\exp\{-\exp(\hat{z}_i)\}]\exp\{-\exp(\hat{z}_i)\}}{\exp[\hat{\alpha}\exp\{-\exp(\hat{z}_i)\}] - 1}, \\
\Delta_{2i} &= \delta_i \hat{v}_i \exp(\hat{z}_i) - \delta_i \hat{v}_i \{\hat{\alpha}\exp[-\exp(\hat{z}_i)]\exp(\hat{z}_i) - 1\} - \frac{\delta_i}{\hat{\sigma}} + \frac{\hat{\alpha}\hat{v}_i\hat{g}_i}{\exp[\hat{\alpha}\exp(\hat{z}_i)] - 1}, \\
\Delta_{ji} &= (\delta_i)\exp(\hat{z}_i)\left(\frac{x_{ij}}{\hat{\sigma}}\right) + (\delta_i)\frac{x_{ij}}{\hat{\sigma}}\left[\hat{\alpha}\exp\{-\exp(\hat{z}_i)\}\exp(\hat{z}_i) - 1\right] \\
&\quad - (1-\delta_i)\frac{\hat{\alpha}\hat{g}_i\left(\frac{x_{ij}}{\hat{\sigma}}\right)}{\exp[\hat{\alpha}\exp\{-\exp(\hat{z}_i)\}] - 1}.
\end{aligned}$$

where $j = 0, 1, \ldots, p$, $i = 1, \ldots, n$, $\hat{g}_i = \exp[\alpha \exp\{-\exp(z_i)\}]\exp\left\{\hat{z}_i - \exp(\hat{z}_i)\right\}$, $\hat{z}_i = (y_i - \mathbf{x}_i^T\hat{\boldsymbol{\beta}})/\hat{\sigma}$ and $\hat{v}_i = (y_i - \mathbf{x}_i^T\hat{\boldsymbol{\beta}})/\hat{\sigma}^2$.

Appendix B. Appendix B: Response perturbation scheme

Here, we provide elements $\mathbf{\Delta}_{ji}$ by considering the response variable perturbation scheme. The elements of matrix $\mathbf{\Delta} = (\mathbf{\Delta}_1^T, \mathbf{\Delta}_2^T, \mathbf{\Delta}_j^T)^T$ are expressed as

$$\begin{aligned}
\Delta_{1i} &= -\delta_i \exp\{-\exp(z_i)\}\exp(z_i)\frac{S_y}{\hat{\sigma}} + (1-\delta_i)\hat{g}_i(S_y/\hat{\sigma})\left[\frac{1 + \hat{\alpha}\exp\{-\exp(\hat{z}_i)\})}{\exp[\hat{\alpha}\exp\{-\exp(\hat{z}_i)\}] - 1}\right] \\
&\quad + (1-\delta_i)\frac{\hat{\alpha}\hat{g}_i^2\{\exp(-\hat{z}_i)\}^2(S_y/\hat{\sigma})}{[\exp[\hat{\alpha}\exp\{-\exp(\hat{z}_i)\}] - 1]^2} \\
\Delta_{2i} &= \delta_i \exp(\hat{z}_i)(S_y/\hat{\sigma}^2) + \delta_i\hat{v}_i\exp(\hat{z}_i)(S_y/\hat{\sigma}) + \delta_i\hat{\alpha}\exp\{-\exp(\hat{z}_i)\}(\exp(z_i))^2(S_y/\hat{\sigma})\hat{v}_i \\
&\quad + \delta_i\hat{\alpha}\exp\{-\exp(\hat{z}_i)\}\exp(\hat{z}_i)(S_y/\hat{\sigma})[1 + \hat{v}_i] - \delta_i(S_y/\hat{\sigma})^2 - (1-\delta_i)\frac{\left\{(S_y/\hat{\sigma}) + (S_y/\hat{\sigma}^2)\right\}\hat{\alpha}\hat{g}_i}{\exp[\alpha\exp(\hat{z}_i)] - 1} \\
&\quad - (1-\delta_i)\frac{\{\hat{\alpha}\exp\{-\exp(\hat{z}_i)\} - \hat{v}_i\exp(\hat{z}_i)\}\hat{\alpha}\hat{g}_i(S_y/\hat{\sigma})}{\exp[\alpha\exp(\hat{z}_i)] - 1}(1-\delta_i)\frac{\hat{\alpha}^2\hat{g}_i^2\hat{v}_i(S_y/\hat{\sigma})}{[\exp[\hat{\alpha}\exp(\hat{z}_i)] - 1]^2}, \\
\Delta_{ji} &= -\delta_i\exp(\hat{z}_i)\left(\frac{x_{ij}}{\hat{\sigma}}\right)\left(\frac{S_y}{\hat{\sigma}}\right) + \delta_i\hat{\alpha}\exp\{-\exp(\hat{z}_i)\}\exp(\hat{z}_i)\left(\frac{x_{ij}}{\hat{\sigma}}\right)\left(\frac{S_y}{\hat{\sigma}}\right)[1 - \exp(\hat{z}_i)] \\
&\quad - (1-\delta_i)\frac{\left[\hat{\alpha}^2\hat{g}_i + \hat{\alpha}\exp(\hat{z}_i)\right]\exp\{-\exp(\hat{z}_i)\}\left(\frac{x_{ij}}{\hat{\sigma}}\right)\exp(\hat{z}_i)\left(\frac{S_y}{\hat{\sigma}}\right)}{\exp[\hat{\alpha}\exp\{-\exp(\hat{z}_i)\}] - 1} - (1-\delta_i)\frac{\hat{\alpha}\hat{g}_i\left(\frac{x_{ij}}{\hat{\sigma}}\right)\left(\frac{S_y}{\hat{\sigma}}\right)}{\exp[\hat{\alpha}\exp\{-\exp(\hat{z}_i)\}] - 1} \\
&\quad - (1-\delta_i)\frac{\hat{\alpha}^2\hat{g}_i^2\left(\frac{x_{ij}}{\hat{\sigma}}\right)\left(\frac{S_y}{\hat{\sigma}}\right)}{[\exp[\hat{\alpha}\exp\{-\exp(\hat{z}_i)\}] - 1]^2},
\end{aligned}$$

where $j = 0, 1, \ldots, p$, $\hat{z}_i = (y_i - \mathbf{x}_i^T\hat{\boldsymbol{\beta}})/\hat{\sigma}$, $\hat{v}_i = (y_i - \mathbf{x}_i^T\hat{\boldsymbol{\beta}})/\hat{\sigma}^2$, $\hat{g}_i = \exp\left\{\hat{z}_i + \exp(\hat{z}_i)\right\}$, $\hat{h}_i = \exp\{\exp(\hat{z}_i)\}$ and $i = 1, \ldots, n$.

Appendix C. Appendix C: Explanatory variable perturbation

Here, we provide elements $\boldsymbol{\Delta}_{ji}$ by considering the explanatory variable perturbation scheme. The elements of matrix $\boldsymbol{\Delta} = (\boldsymbol{\Delta}_1^T, \boldsymbol{\Delta}_2^T, \boldsymbol{\Delta}_j^T)^T$ are expressed as

$$
\begin{aligned}
\Delta_{1i} &= -\delta_i \exp\{-\exp(\hat{z}_i)\}\exp(\hat{z}_i))\,\frac{S_t\beta_t}{\hat{\sigma}} - \frac{\hat{g}_i\frac{S_t\hat{\beta}_t}{\hat{\sigma}}}{\exp\left[\hat{\alpha}\exp\{-\exp(z_i)\}\right]-1} \\
&\quad + (1-\delta_i)\frac{\hat{g}_i\frac{S_t\hat{\beta}_t}{\hat{\sigma}}\exp\{-\exp(z_i)\}}{\exp\left[\hat{\alpha}\exp\{-\exp(z_i)\}\right]-1} - (1-\delta_i)\frac{\hat{\alpha}g_i^2\exp(-\hat{z}_i)\frac{S_t\hat{\beta}_t}{\hat{\sigma}}}{\left[\exp\left[\alpha\exp\{-\exp(\hat{z}_i)\}\right]-1\right]^2}, \\
\Delta_{2i} &= \delta_i\exp(\hat{z}_i)\,\frac{S_t\hat{\beta}_t}{\hat{\sigma}^2}[1+\hat{v}_i^{**}] - \delta_i\hat{\alpha}\frac{S_t\hat{\beta}_t}{\hat{\sigma}}\exp(\hat{z}_i)\exp\{-\exp(\hat{z}_i)\}\left[\exp(\hat{z}_i)\hat{v}_i - 1 + \hat{v}_i\exp(\hat{z}_i)\right] \\
&\quad + \delta_i\frac{S_t\hat{\beta}_t}{\hat{\sigma}^2} - (1-\delta_i)\hat{\alpha}\frac{\exp(\hat{z}_i)\hat{v}_i\frac{S_t\hat{\beta}_t}{\hat{\sigma}}}{\exp\left[\alpha\exp\{-\exp(\hat{z}_i)\}\right]-1}(\hat{v}_i\exp\{-\exp(z_i)\}+1) \\
&\quad + (1-\delta_i)\frac{\hat{\alpha}\hat{g}_i\frac{S_t\hat{\beta}_t}{\hat{\sigma}^2}(v_i+1)}{\exp\left[\alpha\exp\{-\exp(\hat{z}_i)\}\right]-1} + (1-\delta_i)\frac{\hat{\alpha}^2(\hat{g}_i)^2\hat{v}_i\frac{S_t\hat{\beta}_t}{\hat{\sigma}}}{\left[\exp\left[\alpha\exp\{-\exp(z_i)\}\right]-1\right]^2},
\end{aligned}
$$

For $j \neq t$, take the forms

$$
\begin{aligned}
\Delta_{ji} &= \delta_i\exp(z_i)\left(\frac{\hat{\beta}_tS_t}{\hat{\sigma}}\right)\left(\frac{x_{ij}}{\hat{\sigma}}\right) + \delta_i\hat{\alpha}\exp(\hat{z}_i)\exp\{-\exp(z_i)\}\left(\frac{\hat{\beta}_tS_tx_{ij}}{\hat{\sigma}^2}\right)\left[\exp(\hat{z}_i)-1\right] \\
&\quad + (1-\delta_i)\frac{\hat{\alpha}^3\left(\frac{\hat{\beta}_tS_tx_{ij}}{\hat{\sigma}^2}\right)g_i^2}{\exp\left[\alpha\exp\{-\exp(\hat{z}_i)\}\right]-1} + (1-\delta_i)\left(\frac{\hat{\beta}_tS_tx_{ij}}{\hat{\sigma}}\right)\left[\exp\{-\exp(\hat{z}_i)\}\exp(2\hat{z}_i) - \hat{g}_i\right] \\
&\quad - (1-\delta_i)\hat{\alpha}^2\hat{g}_i^2\frac{\left(\frac{\hat{\beta}_tS_tx_{ij}}{\hat{\sigma}^2}\right)}{\left[\exp\left[\hat{\alpha}\exp\{-\exp(\hat{z}_i)\}\right]-1\right]^2}
\end{aligned}
$$

where $j = 0, 1, \ldots, p$, $\hat{z}_i = (y_i - \mathbf{x}_i^T\hat{\boldsymbol{\beta}})/\hat{\sigma}$, $\hat{v}_i = (y_i - \mathbf{x}_i^T\hat{\boldsymbol{\beta}})/\hat{\sigma}^2$, and $i = 1, \ldots, n$. And for $j = t$, take the forms

$$
\begin{aligned}
\Delta_{ti} &= \delta_i\exp(\hat{z}_i)\left(\frac{\hat{\beta}_tS_t}{\hat{\sigma}}\right)\left(\frac{x_{it}}{\hat{\sigma}}\right) + \delta_i\exp(\hat{z}_i)\left(\frac{S_t}{\hat{\sigma}}\right) + \delta_i\hat{\alpha}\exp(2\hat{z}_i)\exp\{-\exp(\hat{z}_i)\}\left(\frac{\hat{\beta}_tS_tx_{it}}{\hat{\sigma}^2}\right) \\
&\quad + \delta_i\hat{\alpha}\exp(\hat{z}_i)\exp\{-\exp(\hat{z}_i)\}\left(\frac{\hat{\beta}_tS_t}{\hat{\sigma}}\right)\left[\frac{S_t}{\hat{\sigma}}-1\right] - \delta_i\frac{S_t}{\hat{\sigma}} - (1-\delta_i)\frac{\hat{\alpha}\hat{g}_i\left(\frac{\hat{\beta}_tS_tx_{ij}}{\hat{\sigma}^2}\right)\left[\exp(-\exp(\hat{z}_i)) + \exp(\hat{z}_i)\right]}{\exp\left[\hat{\alpha}\exp\{-\exp(\hat{z}_i)\}\right]-1} \\
&\quad + (1-\delta_i)\hat{\alpha}\left(\frac{\hat{\beta}_tS_t}{\hat{\sigma}}\right)\hat{g}_i\frac{\left[1+\frac{x_{it}}{\hat{\sigma}}\exp\{-\exp(\hat{z}_i)\}\exp(\hat{z}_i)\right]}{\exp\left[\hat{\alpha}\exp\{-\exp(\hat{z}_i)\}\right]-1} - (1-\delta_i)\hat{\alpha}^3\hat{g}_i^2\frac{\left(\frac{\hat{\beta}_tS_tx_{it}}{\hat{\sigma}^2}\right)}{\left[\exp\left[\hat{\alpha}\exp\{-\exp(\hat{z}_i)\}\right]-1\right]^2}
\end{aligned}
$$

where $\hat{g}_i = \exp\left[\hat{\alpha}\exp\{-\exp(\hat{z}_i)\}\right]\exp\{-\exp(\hat{z}_i)\}\exp(\hat{z}_i)$ and $i = 1, \ldots, n$.