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Extreme-value Modeling
Research Article

A class of multivariate max-infinitely divisible distributions based on random scaling

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Abstract

A class of exchangeable multivariate max-id distributions with ℓ_1-norm symmetric exponent measure was introduced by Genest, Nešlehová and Rivest in a paper published in the journal *Bernoulli* in 2018. Three years later, an extended class of exchangeable multivariate max-id distributions with ℓ_p-norm symmetric exponent measure was proposed by Mai and Wang in an article which appeared in the *Journal of Multivariate Analysis.* A new class is proposed here which encompasses them both and which allows for non-exchangeability, thereby providing extra flexibility for modeling multivariate block maxima data and extreme risks. Some properties of members of this class are studied, and conditions are given under which they are multivariate extreme-value distributions. The maximum attractor of each class member is also determined under broad conditions, and an algorithm due to Jan-Frederik Mai is adapted for simulation purposes.

Keywords: Exponent measure · Extreme-value distribution · Maximum domain of attraction · Multivariate stochastic model · Simulation algorithm

Mathematics Subject Classification: Primary 60E07 · Secondary 60G07, 62H05

1. Introduction

In recent years, many models have been proposed for the analysis of extreme risks based on max-stable distributions or, more generally, max-stable spatial processes; see, for example, [1]. These distributions, which arise as weak limits of normalized component-wise maxima of mutually independent and identically distributed random vectors or processes, provide an asymptotic approximation to reality. However, as recently discussed in the position paper by Huser et al. [2], for example, this approximation may be too crude when dealing with maxima over finitely many observations, particularly in environmental studies.

Max-infinitely divisible distributions (max-id for short) constitute a substantially wider class of models that encompasses and expands the set of max-stable distributions, thereby providing greater flexibility than the latter while retaining some of their attractive theoretical properties. Specifically, a multivariate distribution function F is called max-id if F^t is a distribution function for any scalar $t \in (0, \infty)$. Equivalently, F is max-id if and only if for each integer $n \in \mathbb{N} = \{1, 2, \dots\}$, there exists a distribution F_n such that $F = F_n^n$. More-

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over, these distributions arise as limits of component-wise maxima of triangular arrays [3]. For examples and discussion, see, for example, [4].

One of the advantages of max-id distributions pointed out in the literature is their ability to capture residual dependence between extreme risks that are asymptotically independent, that is, in the domain of attraction of a max-stable distribution with mutually independent margins. Specific max-id models that can achieve this goal have been proposed in [5] and [6], among others, where they were used to analyze extreme precipitation and wind gusts, respectively. Simulation of max-id processes has also been addressed in [7] and [8].

This article proposes and investigates a class of max-id distributions on the positive orthant that extends the work in [9, 10, 11]. Recall from Proposition 5.8 in [12] that any max-id random vector $\boldsymbol{Y} = (Y_1, \ldots, Y_d)$ with distribution function F such that $\inf\{\boldsymbol{x} \in \mathbb{R}^d \colon F(\boldsymbol{x}) > 0\} = \boldsymbol{0} = (0, \ldots, 0)$ is distributed as the component-wise maximum

$$\max\{\boldsymbol{0}, \max(\boldsymbol{Z}_n \colon n \in \mathcal{N})\}, \tag{1.1}$$

where $\boldsymbol{Z}_n$ is the nth element in a possibly empty but at most countable set $\mathcal{N}$ of points from a Poisson point process on the punctured set $E_d = [0, \infty]^d \setminus \{\boldsymbol{0}\}$ with intensity μ, which is an exponent measure on E_d. That is, μ is a Radon measure on E_d and

$$\mu\left[\bigcup_{j=1}^{d}\{(y_1, \ldots, y_d) \in E_d \colon y_j = \infty\}\right] = 0. \tag{1.2}$$

In (1.1), the value $\boldsymbol{Y} = \boldsymbol{0}$ arises when the set $\mathcal{N}$ is empty. Basic facts about max-id distributions are summarized in Chapter 5 of [12].

The starting point of the present investigation is the class of max-id distributions introduced in [9], each of which has an ℓ_1-norm symmetric exponent measure. To describe the elements of this class, let $\|\cdot\|_1$ represent the ℓ_1-norm on $\mathbb{R}^d$, denote the unit simplex by

$$\mathcal{S}_d = \{\boldsymbol{x} = (x_1, \ldots, x_d) \in [0, 1]^d \colon \|\boldsymbol{x}\|_1 = x_1 + \cdots + x_d = 1\},$$

and let $\mathcal{T} \colon E_d \to (0, \infty] \times \mathcal{S}_d$ be the polar coordinate transformation defined by

$$\mathcal{T}(\boldsymbol{x}) = \left(\|\boldsymbol{x}\|_1, \frac{x_1}{\|\boldsymbol{x}\|_1}, \ldots, \frac{x_d}{\|\boldsymbol{x}\|_1}\right).$$

for all $\boldsymbol{x} = (x_1, \ldots, x_d) \in E_d$. By Theorem 2 in [9], μ is an ℓ_1-norm symmetric exponent measure if and only if its image by $\mathcal{T}$ is of the form $\nu \otimes \upsilon_d$, where υ_d is the uniform distribution on $\mathcal{S}_d$ and the so-called radial measure ν is Radon on $(0, \infty]$ with $\nu\{\infty\} = 0$.

Let $\mathcal{M}_1(\upsilon_d)$ denote the above class of max-id random vectors with ℓ_1-norm symmetric exponent measure. As shown in [9], members of $\mathcal{M}_1(\upsilon_d)$ with $\nu(0, \infty] = \infty$ have a unique copula which is reciprocal Archimedean, and vice versa. An exact method for simulating data from elements of $\mathcal{M}_1(\upsilon_d)$ was devised by Mai [10] as an extension of an algorithm due to Dombry et al. [13]. As pointed out in Remark 2.4 of [10], this procedure also works when υ_d is replaced by an arbitrary probability measure σ_d on $\mathcal{S}_d$. However, the corresponding extended class $\mathcal{M}_1(\sigma_d)$ of distributions has hitherto not been further studied.

The specific objective of this article is to investigate the class $\mathcal{M}_1(\sigma_d)$ and an extension thereof which provides additional modeling flexibility. Specifically, a class $\mathcal{M}_\varrho(\sigma_d)$ of distributions indexed by a parameter $\varrho \in (0, \infty)$ is obtained upon replacing $\mathcal{T}$ by the transformation $\mathcal{T}_\varrho$: $E_d \to (0,\infty] \times \mathcal{S}_d$ defined, for all vectors $\boldsymbol{x} = (x_1, \ldots, x_d) \in E_d$, by

$$\mathcal{T}_\varrho(\boldsymbol{x}) = \left(\|\boldsymbol{x}\|_\varrho, \frac{x_1^\varrho}{\|\boldsymbol{x}\|_\varrho^\varrho}, \ldots, \frac{x_d^\varrho}{\|\boldsymbol{x}\|_\varrho^\varrho} \right), \tag{1.3}$$

where $\|\boldsymbol{x}\|_\varrho = (x_1^\varrho + \cdots + x_d^\varrho)^{1/\varrho}$ refers to the ℓ_ϱ-quasinorm of the vector $\boldsymbol{x}$, which is also the ℓ_ϱ-norm of $\boldsymbol{x}$ when $\varrho \in [1,\infty)$. While all elements of the class $\mathcal{M}_\varrho(\sigma_d)$ are max-id, exchangeability occurs only if σ_d is, which enhances their practical relevance.

A formal definition and some basic properties of the class $\mathcal{M}_\varrho(\sigma_d)$ are given in Section 2, starting with the fact that its members are in one-to-one correspondence with the set of Radon measures ν on $(0,\infty]$ with $\nu\{\infty\} = 0$ or, alternatively, with the corresponding set of generalized survival functions S_ν defined, for every real $t \in (0,\infty)$, by $S_\nu(t) = \nu(t,\infty]$. Section 3 covers the case in which σ_d is a Dirichlet distribution. In particular when $\sigma_d = \upsilon_d$ is uniformly distributed on the simplex, the class $\mathcal{M}_\varrho(\upsilon_d)$ with fixed $\varrho \in [1,\infty)$ is related to the class of max-id distributions with ℓ_ϱ-norm symmetric exponent measure due to [11].

Next, generalized survival functions of specific forms are considered. In Section 4, the generalized survival function S_ν is assumed to be proportional to the map $t \mapsto t^{-\theta}$ for some scalar $\theta \in (0,\infty)$. This allows for an exploration of the intersection between the class $\mathcal{M}_\varrho(\sigma_d)$ and the set of multivariate max-stable distributions. In Section 5, the maximum attractor of elements in the class $\mathcal{M}_\varrho(\sigma_d)$ is determined in the special case of a regularly varying generalized survival function S_ν.

In Section 6, it is shown that for members of the class $\mathcal{M}_\varrho(\sigma_d)$, the random point $\boldsymbol{Z}_n$ in the representation (1.1) is of the form $R_n \boldsymbol{Q}_n{}^{1/\varrho}$, where $\mathcal{R} = \{R_n\colon n \in \mathcal{N}\}$ is a possibly empty but at most countable set of points from a Poisson point process with intensity ν and the random vectors $\boldsymbol{Q}_n$ are mutually independent, distributed as σ_d, and independent of all the elements in $\mathcal{R}$. In general, stochastic representations of multivariate distributions are desirable because they shed light on their nature and properties; this is illustrated, for example, by the work of Wolf-Dieter Richter and his collaborators in this journal [14, 15, 16]. Here, the representation leads to a simulation algorithm anticipated in [10]. It also reveals connections with some of the models in [6], as discussed in Section 7. All proofs are relegated to the Appendix.

Throughout this paper, vectors in $\mathbb{R}^d$ are denoted by boldface symbols; for example, $\boldsymbol{x} = (x_1, \ldots, x_d)$, $\boldsymbol{y} = (y_1, \ldots, y_d)$, $\boldsymbol{0} = (0, \ldots, 0)$, and $\boldsymbol{1} = (1, \ldots, 1)$. By convention, any operation or map applied to vectors is understood component-wise; notably, for any scalar $\eta \in \mathbb{R}$, $\boldsymbol{x}^\eta = (x_1^\eta, \ldots, x_d^\eta)$. Moreover, $\min(\boldsymbol{x}) = \min\{x_1, \ldots, x_d\}$ and $\max(\boldsymbol{y}) = \max\{y_1, \ldots, y_d\}$ for any vectors $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^d$. Analogous notation is used for random vectors.

2. Definitions and basic properties

This section presents a formal definition and some elementary properties of the new class $\mathcal{M}_\varrho(\sigma_d)$ of asymmetric multivariate max-id distributions.

Definition 2.1 Given a scalar $\varrho \in (0,\infty)$ and a probability measure σ_d on the simplex $\mathcal{S}_d$, a random vector $\boldsymbol{Y}$ supported on $[0,\infty)^d$ is said to belong to the class $\mathcal{M}_\varrho(\sigma_d)$ if and only if it is max-id and the image of its exponent measure μ by the transformation $\mathcal{T}_\varrho$ in (1.3) is of the form $\nu \otimes \sigma_d$.

Remark 2.2 Two choices in Definition 2.1 warrant explanation:

(i) First, one may wonder why the map $\mathcal{T}_\varrho$ was used instead of the more intuitive polar coordinate decomposition with respect to the ℓ_ϱ-quasinorm $\|\cdot\|_\varrho$, viz.

$$\mathcal{T}_\varrho^*\colon \boldsymbol{x} \mapsto \left(\|\boldsymbol{x}\|_\varrho, \frac{x_1}{\|\boldsymbol{x}\|_\varrho}, \ldots, \frac{x_d}{\|\boldsymbol{x}\|_\varrho} \right),$$

defined for any vector $\boldsymbol{x} = (x_1, \ldots, x_d) \in E_d$. One can easily check that the image of μ by $\mathcal{T}_\varrho$ is of the form $\nu \otimes \sigma_d$ if and only if the image of μ by $\mathcal{T}_\varrho^*$ is of the form $\nu \otimes \sigma_d^*$, where σ_d^* and σ_d are related through $\sigma_d(A) = \sigma_d^*\{\boldsymbol{x}^\varrho\colon \boldsymbol{x} \in A\}$ for any Borel set $A \subseteq \mathcal{S}_d$. Thus the map $\mathcal{T}_\varrho^*$ could have been used in Definition 2.1 without altering the class of max-id distributions under study. Working with $\mathcal{T}_\varrho$ has the advantage that it maps E_d to $(0, \infty] \times \mathcal{S}_d$ irrespective of ϱ. This simplifies certain calculations; see, for instance, the proof of Theorem 3.6.

(ii) Second, the condition that σ_d is a probability measure is imposed for identifiability purposes; this measure could have been assumed to be merely finite without altering the class of distributions. Indeed, the measure $\nu^* \otimes \sigma_d^*$ with σ_d^* such that $s = \sigma_d^*(\mathcal{S}_d) \in (0, \infty)$ is the same as the measure $\nu \otimes \sigma_d$, where $\nu = s\nu^*$ and $\sigma_d = \sigma_d^*/s$ is a probability measure. Because σ_d is a probability measure, random vectors with law σ_d can be employed—a convenience that will become evident subsequently.

The condition that σ_d is a probability measure in Definition 2.1 leads to a one-to-one correspondence between the class $\mathcal{M}_\varrho(\sigma_d)$ and the set of radial measures. This is stated below as Lemma 2.4. For completeness, the notion of radial measure is recalled first.

Definition 2.3 A Radon measure ν on $(0, \infty]$ with $\nu\{\infty\} = 0$ is called a radial measure. Furthermore, the generalized survival function $S_\nu\colon (0, \infty) \to [0, \infty)$ of ν is defined by $S_\nu(t) = \nu(t, \infty]$ for every real $t \in (0, \infty)$.

As noted in [9] and [10], one has $S_\nu(t) < \infty$ for every real $t \in (0, \infty)$ because ν is Radon, and $\nu\{\infty\} = 0$ implies that $S_\nu(t) \to 0$ as $t \to \infty$. Note also that S_ν is right-continuous and non-increasing with $S_\nu(t) \to \nu(0, \infty]$ as $t \to 0$.

Lemma 2.4 Let σ_d be an arbitrary probability measure on $\mathcal{S}_d$.

(i) If μ is an exponent measure on E_d whose image by $\mathcal{T}_\varrho$ is of the form $\nu \otimes \sigma_d$, then ν is a radial measure.

(ii) If ν is a radial measure, then the image measure μ of $\nu \otimes \sigma_d$ by $\mathcal{T}_\varrho^{-1}$ is an exponent measure on E_d.

The result below, part of which is stated without proof in Remark 2.4 of [10], gives expressions for the distribution function of any member of $\mathcal{M}_\varrho(\sigma_d)$.

Proposition 2.5 Let $\boldsymbol{Y}$ be a random vector from the class $\mathcal{M}_\varrho(\sigma_d)$ with radial measure ν. Let also $\boldsymbol{Q}$ be a random vector with distribution σ_d on $\mathcal{S}_d$. Then, for every $\boldsymbol{y} \in [0, \infty)^d$,

$$\Pr(\boldsymbol{Y} \leq \boldsymbol{y}) = \exp\Big[-\mathrm{E}\big[S_\nu\{\min(\boldsymbol{y}/\boldsymbol{Q}^{1/\varrho})\}\big]\Big]. \tag{2.1}$$

Furthermore, if H is the distribution function of $\boldsymbol{Q}$ and $\bar{H}$ denotes the corresponding survival function, then for every $\boldsymbol{y} \in (0, \infty)^d$, one has both

$$\Pr(\boldsymbol{Y} \leq \boldsymbol{y}) = \exp\left[-\int_0^\infty \{\mathbf{1} - H(\boldsymbol{y}^\varrho/r^\varrho)\}\, \mathrm{d}\nu(r)\right]$$

and

$$\Pr(\boldsymbol{Y} \le \boldsymbol{y}) = \exp\left\{ \sum_{k=1}^{d} (-1)^k \sum_{A \subseteq \{1,\ldots,d\},\, |A|=k} \int_{\max(\boldsymbol{y}_A)}^{\infty} \bar{H}(\boldsymbol{y}_A^{\varrho}/r^{\varrho})\, \mathrm{d}\nu(r) \right\},$$

where the jth component of $\boldsymbol{y}_A$ is given by y_j if $j \in A$ and by 0 otherwise.

Proposition 2.5 implies that for every integer $j \in \{1, \ldots, d\}$, the jth univariate margin F_j of a random vector $\boldsymbol{Y}$ from the class $\mathcal{M}_\varrho(\sigma_d)$ satisfies, for every real $y \in (0, \infty)$,

$$F_j(y) = \exp\bigl[-\mathrm{E}\bigl\{S_\nu(y/Q_j^{1/\varrho})\bigr\}\bigr] = \exp\left\{ -\int_0^\infty \bar{H}_j(y^\varrho/r^\varrho)\, \mathrm{d}\nu(r) \right\}, \tag{2.2}$$

where $\bar{H}_j$ denotes the survival function of Q_j.

Moreover, $F_j(0) = \Pr(Y_j \le 0) = \exp\{-\Pr(Q_j > 0)\nu(0, \infty]\}$, which shows that Y_j has an atom at 0 if the measure ν is finite. Further note that F_j is continuous on $(0, \infty)$ if either S_ν or $\bar{H}_j$ is continuous on that set. In addition, owing to the fact that $S_\nu(\infty) = 0$, observe that if $Q_j = 0$ almost surely, then Y_j is degenerate, that is,

$$\forall_{y \in (0,\infty)}\ F_j(y) = 1. \tag{2.3}$$

The next result clarifies the role of the parameter ϱ and its impact on the dependence between the components of $\boldsymbol{Y}$. Recall that if all the univariate margins of this vector are continuous, the dependence structure of $\boldsymbol{Y}$ is then characterized by a unique copula through Sklar's representation theorem; see, for example, [17, 18] or Theorem 2.3.3 in [19].

Lemma 2.6 Let $\varrho, \varrho^* \in (0, \infty)$ be distinct scalars and $\boldsymbol{Y}$ be a random vector from the class $\mathcal{M}_\varrho(\sigma_d)$ with radial measure ν. Let also $\boldsymbol{Z}$ be a random vector in $\mathcal{M}_{\varrho^*}(\sigma_d)$ with radial measure ν_{ϱ,ϱ^*}, whose generalized survival function is given by $S_{\nu_{\varrho,\varrho^*}}(t) = S_\nu(t^{\varrho^*/\varrho})$ for every real $t \in (0, \infty)$. Then, the following statements hold true.

(i) $\boldsymbol{Y}$ has the same distribution as $\boldsymbol{Z}^{\varrho^*/\varrho}$.
(ii) If $\boldsymbol{Y}$ is continuous, then $\boldsymbol{Y}$ and $\boldsymbol{Z}$ have the same (unique) copula.

Lemma 2.6 implies that if the focus is on copulas, one can consider the class $\mathcal{M}_1(\sigma_d)$ without loss of generality. Nevertheless, this does not imply that ϱ plays no role in shaping the copula. Indeed, this parameter features in the radial measure of $\boldsymbol{Z}$ in Lemma 2.6 and impacts the dependence structure; this is portrayed later in Figure 1.

Elements of the new class of asymmetric multivariate max-id distributions can only model positive association, as was the case for reciprocal Archimedean copulas in [9], which they extend. This is because all max-id distributions are multivariate totally positive of order 2 (MTP_2), as shown in [20]. A multivariate distribution function F is said to be MTP_2 if and only if, for any vectors $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^d$, one has $F\{\max(\boldsymbol{x}, \boldsymbol{y})\}F\{\min(\boldsymbol{x}, \boldsymbol{y})\} \ge F(\boldsymbol{x})F(\boldsymbol{y})$, where the operations max and min are meant component-wise. Further note that the two limiting cases of positive association, namely independence and comonotonicity, can be achieved through specific choices of probability measure σ_d, as stated below.

Proposition 2.7 The following statements hold true.

(i) Suppose that the probability measure σ_d is discrete with support $\{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_d\}$, where for each integer $j \in \{1, \ldots, d\}$, $\boldsymbol{e}_j$ denotes the d-variate vector whose components are all equal to 0 except the jth, which is equal to 1. Then any random vector $\boldsymbol{Y}$ from the class $\mathcal{M}_\varrho(\sigma_d)$ has mutually independent components.

(ii) Let $A_1, \ldots, A_K$ be a partition of the set $\{1, \ldots, d\}$. For each integer $k \in \{1, \ldots, K\}$, let σ_{A_k} be a probability distribution on $\mathcal{S}_d$ with the property that $\sigma_{A_k}\{\boldsymbol{s} = (s_1, \ldots, s_d) \in \mathcal{S}_d \colon \forall_{j \notin A_k} s_j = 0\} = 1$. Suppose that $\sigma_d = p_1\sigma_{A_1} + \cdots + p_K\sigma_{A_K}$ for some scalars $p_1, \ldots, p_K \in (0, 1)$ such that $p_1 + \cdots + p_K = 1$. Then for any random vector $\boldsymbol{Y} \in \mathcal{M}_\varrho(\sigma_d)$, the subvectors $\boldsymbol{Y}_{A_1} = (Y_j \colon j \in A_1), \ldots, \boldsymbol{Y}_{A_K} = (Y_j \colon j \in A_K)$ are mutually independent.

(iii) Suppose that the probability measure σ_d is degenerate and places all its mass on some vector $\boldsymbol{q} \in \mathcal{S}_d$ with $q_j \neq 0$ for every integer $j \in \{1, \ldots, d\}$. Then any random vector $\boldsymbol{Y}$ from the class $\mathcal{M}_\varrho(\sigma_d)$ is comonotone.

In Sections 3 to 5, subclasses of models are considered which correspond to specific choices of probability measure σ_d and survival function S_ν.

3. The case in which σ_d is Dirichlet

The family $\{\Delta_{\boldsymbol{\alpha}} \colon \boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_d) \in (0, \infty)^d\}$ of Dirichlet distributions is a well-known class of probability laws on $\mathcal{S}_d$ which is both rich and tractable. For this reason, it constitutes a natural choice for σ_d, which is investigated below.

When $\boldsymbol{\alpha} = \mathbf{1}$, the Dirichlet distribution boils down to the uniform distribution υ_d on $\mathcal{S}_d$. Unless $\alpha_1 = \cdots = \alpha_d$, $\Delta_{\boldsymbol{\alpha}}$ is not exchangeable, and this asymmetry propagates to random vectors in the class $\mathcal{M}\varrho(\Delta\boldsymbol{\alpha})$. This provides additional modeling flexibility, as mentioned in the Introduction; see Figure 1 for an illustration.

Apart from the extra modeling flexibility, the choice $\sigma_d = \Delta_{\boldsymbol{\alpha}}$ induces a family of max-id distributions which are stable with respect to marginalization, as will be shown next. To fix ideas, consider an arbitrary random vector $\boldsymbol{Y}$ from the class $\mathcal{M}_\varrho(\sigma_d)$ and focus on the subvector $\boldsymbol{Y}_k = (Y_1, \ldots, Y_k)$ for some integer $k \in \{2, \ldots, d-1\}$, without loss of generality. From Proposition 2.5, one has

$$\Pr(Y_1 \le y_1, \ldots, Y_k \le y_k) = \exp\Big[-\mathrm{E}\big[S_\nu\{\min(y_1/Q_1^{1/\varrho}, \ldots, y_k/Q_k^{1/\varrho})\}\big]\Big]$$

for all $y_1, \ldots, y_k \in [0, \infty)$. While this expression resembles (2.1), it does not guarantee that $\boldsymbol{Y}_k \in \mathcal{M}_\varrho(\sigma_k^*)$ for some probability measure σ_k^* on the k-dimensional simplex $\mathcal{S}_k$. This is because $(Q_1, \ldots, Q_k)$ is not supported on the latter unless $Q_{k+1} = \cdots = Q_d = 0$ almost surely. Nevertheless, the next result shows that for certain distributions on $\mathcal{S}_d$ that do not concentrate on facets, including the Dirichlet distribution, one still has $\boldsymbol{Y}_k \in \mathcal{M}_\varrho(\sigma_k^*)$.

Proposition 3.1 Consider a random vector $\boldsymbol{Y}$ from the class $\mathcal{M}_\varrho(\sigma_d)$ with radial measure ν and let $\boldsymbol{Y}_\mathcal{I} = (Y_i \colon i \in \mathcal{I})$ denote the marginal subvector of $\boldsymbol{Y}$ with indices in $\mathcal{I} \subsetneq \{1, \ldots, d\}$. Let also $\boldsymbol{Q}$ be a random vector with distribution σ_d and set $\boldsymbol{Q}_\mathcal{I} = (Q_i \colon i \in \mathcal{I})$. Therefore, define $W_\mathcal{I} = \sum_{i \in \mathcal{I}} Q_i$ and denote its distribution function by $F_\mathcal{I}$. Suppose that $W_\mathcal{I} > 0$ almost surely and that $W_\mathcal{I}$ is independent of $\boldsymbol{Q}_\mathcal{I}^* = \boldsymbol{Q}_\mathcal{I}/W_\mathcal{I}$. Then, $\boldsymbol{Y}_k \in \mathcal{M}_\varrho(\sigma_\mathcal{I}^*)$ with radial measure $\nu_\mathcal{I}^*$, where $\sigma_\mathcal{I}^*$ is the distribution of $\boldsymbol{Q}_\mathcal{I}^*$ and $\nu_\mathcal{I}^*$ has survival function given, for all $t \in (0, \infty)$, by

$$S_{\nu_\mathcal{I}^*}(t) = \int_0^1 S_\nu(t/w^{1/\varrho}) \, \mathrm{d}F_\mathcal{I}(w).$$

When the random vector $\boldsymbol{Q}$ has distribution $\Delta_{\boldsymbol{\alpha}}$, it is well known that for any proper subset $\mathcal{I}$ of $\{1, \ldots, d\}$, $\boldsymbol{Q}_\mathcal{I}^*$ is independent of $W_\mathcal{I}$. Moreover, $W_\mathcal{I}$ has a Beta distribution $\mathcal{B}(\sum_{i \in \mathcal{I}} \alpha_i, \sum_{i \notin \mathcal{I}} \alpha_i)$, while $\boldsymbol{Q}_\mathcal{I}^*$ is again Dirichlet with parameter vector $\boldsymbol{\alpha}_\mathcal{I} = (\alpha_i \colon i \in \mathcal{I})$. Therefore, $\boldsymbol{Y}_\mathcal{I} \in \mathcal{M}_\varrho(\Delta_{\boldsymbol{\alpha}_\mathcal{I}})$ for any set $\mathcal{I} \subsetneq \{1, \ldots, d\}$ with cardinality $|\mathcal{I}| \ge 2$.

Next, consider the dependence structure of an arbitrary random vector $\boldsymbol{Y}$ in the class $\mathcal{M}_\varrho(\Delta_{\boldsymbol{\alpha}})$. To this end, first observe that if $\nu(0,\infty] = \infty$, the univariate margins of $\boldsymbol{Y}$ are continuous in view of the discussion following (2.2) and the fact that the univariate margins of the Dirichlet distribution $\Delta_{\boldsymbol{\alpha}}$ are Beta. The max-id random vector $\boldsymbol{Y}$ thus has a unique copula, say $C^{\nu}_{\varrho,\boldsymbol{\alpha}}$, as soon as $\nu(0,\infty] = \infty$. In view of Lemma 2.6, $C^{\nu}_{\varrho,\boldsymbol{\alpha}}$ coincides with $C^{\nu_{\varrho,1}}_{1,\boldsymbol{\alpha}}$, so that one can set $\varrho = 1$ without loss of generality when focusing on the dependence structure only. These considerations lead to the following definition.

Definition 3.2 A d-variate copula with parameter $\boldsymbol{\alpha} \in (0,\infty)^d$ and unbounded radial measure ν is said to be a reciprocal Liouville copula, denoted $C^{\nu}_{1,\boldsymbol{\alpha}}$, if and only if it is the unique copula of a max-id random vector $\boldsymbol{Y}$ in the class $\mathcal{M}_1(\Delta_{\boldsymbol{\alpha}})$ with radial measure ν.

The reasons justifying the choice of the term "reciprocal Liouville" are threefold:

(i) As already mentioned in the Introduction, the special case $\alpha = \mathbf{1}$ corresponds to the reciprocal Archimedean copulas in [9].

(ii) The extension from the uniform distribution υ_d on $\mathcal{S}_d$ to a general Dirichlet distribution $\Delta_{\boldsymbol{\alpha}}$ evokes a similar use of the Dirichlet distribution in the stochastic representation of Archimedean copulas to create the so-called Liouville copulas introduced in [21]; examples presented later in this article will indeed reveal certain similarities between the Liouville and reciprocal Liouville copulas.

(iii) The points arising from the Poisson point process in the representation of Section 6 are reminiscent of the structure of Liouville random vectors, which are random scale mixtures of the Dirichlet distribution; for background on the Dirichlet and Liouville distributions, see [22].

Results to be presented in subsequent sections of this article will lead to several properties of reciprocal Liouville copulas, along with an algorithm for random number generation; see Examples 5.2 and 6.2, as well as Corollary 5.4.

The remainder of this section is devoted to the special case in which $\Delta_{\boldsymbol{\alpha}} = \upsilon_d$, in order to relate the class $\mathcal{M}_\varrho(\upsilon_d)$ to various multivariate models that have been proposed in the literature. To see this, and to understand the properties of the class $\mathcal{M}_\varrho(\upsilon_d)$, it is convenient to recall that an exponent measure μ on E_d with the additional property that

$$\mu\big\{\boldsymbol{x} = (x_1,\ldots,x_d) \in E_d\colon \exists_{k\in\{1,\ldots,d\}}\ x_k = 0\big\} = 0 \tag{3.1}$$

is uniquely determined through its generalized survival function given, for every vector $\boldsymbol{x} \in E_d$, by $S_\mu(\boldsymbol{x}) = \mu(\boldsymbol{x},\boldsymbol{\infty}]$; see Lemma 5 in [9].

Definition 3.3 An exponent measure μ on E_d is called ℓ_ϱ-quasinorm symmetric for some scalar $\varrho \in (0,\infty)$ if (3.1) holds and if there exists a map Λ: $(0,\infty) \to [0,\infty)$ with $\Lambda(t) \to 0 \equiv \Lambda(\infty)$ as $t \to \infty$ such that, for every vector $\boldsymbol{x} \in E_d$, $S_\mu(\boldsymbol{x}) = \Lambda(\|\boldsymbol{x}\|_\varrho^\varrho)$.

The map Λ is unique and called the generator of μ and of the max-id random vector with exponent measure μ.

Remark 3.4 When $\varrho = 1$, the above definition coincides with that of an ℓ_1-norm symmetric exponent measure from Definition 4 in [9]. The case $\varrho = p \in (1,\infty)$ corresponds to the ℓ_p-norm symmetric exponent measure from Section 5 of [11], although condition (3.1) was not stated explicitly and their definition used a map φ defined, for every real $t \in (0,\infty)$, by $\varphi(t) = \Lambda(t^\varrho)$. However, as noted in [23] and seen below, working with Λ is more convenient.

The following result describes the necessary and sufficient conditions required for a real-valued function defined on the interval $(0,\infty)$ to be the generator of an ℓ_ϱ-quasinorm symmetric exponent measure.

Proposition 3.5 The map Λ: $(0,\infty) \to [0,\infty)$ is the generator of an ℓ_ϱ-quasinorm symmetric exponent measure if and only if $\Lambda(t) \to 0$ as $t \to \infty$ and Λ is d-monotone on $(0,\infty)$. The latter condition means that:

(i) Λ is differentiable on $(0,\infty)$ up to the order $d-2$;
(ii) for every integer $k \in \{0,\dots,d-2\}$, the kth derivative of Λ satisfies $(-1)^k\Lambda^{(k)}(t) \ge 0$ for every real $t \in (0,\infty)$;
(ii) $(-1)^{d-2}\Lambda^{(d-2)}$ is non-increasing and convex on $(0,\infty)$.

Next, it will be shown that the class of max-id random vectors having an ℓ_ϱ-quasinorm symmetric exponent measure coincides with the class $\mathcal{M}_\varrho(\upsilon_d)$. The following result, which relies on the notion of Williamson transform [24, 25], specifies the one-to-one relationship between the generator and the radial measure, in analogy with Theorem 2 in [9].

Theorem 3.6 An exponent measure μ on E_d is ℓ_ϱ-quasinorm symmetric with generator Λ if and only if the image of μ by the transformation $\mathcal{T}_\varrho$ is of the form $\nu\otimes\upsilon_d$ for some radial measure ν. Moreover, there exists a bijective relationship between ν and Λ, which can be expressed through the generalized survival function S_ν associated with ν, as follows.

(i) Λ is the Williamson d-transform of $\nu_{\varrho,1}$ given, for every real $t \in (0,\infty)$, by

$$\Lambda(t) = \mathfrak{W}_d(\nu_{\varrho,1})(t) = \int_0^\infty (1-t/r)_+^{d-1}\,\mathrm{d}\nu_{\varrho,1}(r),$$

where $t_+ = \max(t,0)$ and $\nu_{\varrho,1}$ is as in Lemma 2.6 with $\varrho^* = 1$.

(ii) $S_\nu(t) = \mathfrak{W}_d^{-1}(\Lambda)(t^\varrho)$ for every real $t \in (0,\infty)$, where $\mathfrak{W}_d^{-1}(\Lambda)$ is the inverse Williamson d-transform of Λ, defined for every real $r \in (0,\infty)$, by

$$\mathfrak{W}_d^{-1}(\Lambda)(r) = \sum_{k=0}^{d-2}\frac{(-1)^k\Lambda^{(k)}(r)}{k!}\,r^k + \frac{(-1)^{d-1}\Lambda_+^{(d-1)}(r)}{(d-1)!}\,r^{d-1},$$

where $\Lambda^{(k)}$ (respectively $\Lambda_+^{(k)}$) is the kth order (right-hand) derivative of Λ.

The following corollary to Theorem 3.6 is a consequence of Lemma 2.6.

Corollary 3.7 If a random vector $\boldsymbol{Y}$ in the class $\mathcal{M}_\varrho(\upsilon_d)$ has generator Λ such that $\Lambda(t) \to \infty$ as $t \to 0$, its univariate margins are continuous and its copula is reciprocal Archimedean with generator $F = \exp(-\Lambda)$, as defined in expression (3) in [9].

4. The case in which S_ν is of power type

Suppose that the generalized survival function S_ν of a radial measure ν is given, for every real $t \in (0,\infty)$, by $S_\nu(t) = \gamma\, t^{-\theta}$ for some scalars $\gamma, \theta \in (0,\infty)$. It will be shown below that in this case, the corresponding random vector $\boldsymbol{Y}$ from the class $\mathcal{M}_\varrho(\sigma_d)$ has a multivariate extreme-value distribution, provided that its univariate margins are not degenerate.

Before formally stating and proving this result, recall that a multivariate extreme-value distribution is specified through:

(i) its univariate margins, which are generalized extreme-value;
(ii) its underlying copula C, given, for all reals $u_1,\dots,u_d \in (0,1)$, by $C(u_1,\dots,u_d) = \exp\big[-\ell\{|\ln(u_1)|,\dots,|\ln(u_d)|\}\big]$, where ℓ: $[0,\infty)^d \mapsto \mathbb{R}$ is the so-called stable tail dependence function [26, 27].

Moreover, the stable tail dependence function is uniquely specified through its spectral (or angular) measure ς_d, which is a probability distribution on $\mathcal{S}_d$ with the property that for any random vector $\boldsymbol{W} = (W_1, \ldots, W_d)$ with distribution ς_d, one has $\mathrm{E}(W_1) = \cdots = \mathrm{E}(W_d) = 1/d$. Then, for all reals $x_1, \ldots, x_d \in [0, \infty)$,

$$\ell(x_1, \ldots, x_d) = d\mathrm{E}\left\{\max_{j\in\{1,\ldots,d\}}(x_j W_j)\right\}.$$

Proposition 4.1 Suppose that ν is a radial measure with generalized survival function given, for every real $t \in (0, \infty)$, by $S_\nu(t) = \gamma\, t^{-\theta}$ for some scalars $\gamma,\ \theta \in (0, \infty)$. Let $\boldsymbol{Q} = (Q_1, \ldots, Q_d)$ be a random vector with distribution σ_d on $\mathcal{S}_d$ such that $Q_1 \not\equiv 0, \ldots, Q_d \not\equiv 0$ almost surely. Then the random vector $\boldsymbol{Y} = (Y_1, \ldots, Y_d)$ from the class $\mathcal{M}_\varrho(\sigma_d)$ with radial measure ν has a multivariate extreme-value distribution. Moreover, for every integer $j \in \{1, \ldots, d\}$, Y_j has a Fréchet distribution with location parameter $a_j = 0$, scaling parameter $b_j = \{\gamma\, \mathrm{E}(Q_j^{\theta/\varrho})\}^{1/\theta}$, and shape parameter θ.

The condition that $Q_1 \not\equiv 0, \ldots, Q_d \not\equiv 0$ almost surely in the above result ensures that the univariate margins of $\boldsymbol{Y}$ are not degenerate. Indeed, whatever the integer $j \in \{1, \ldots, d\}$, the distribution function F_j of Y_j is continuous except possibly at 0 because the map defined by $S_\nu(t) = \gamma\, t^{-\theta}$ for all $t \in (0, \infty)$ is continuous on its entire domain. Moreover, in view of (2.2) and the fact that $\nu(0, \infty] = \infty$, Y_j has an atom at 0 if and only if $Q_j = 0$ almost surely, that is, if σ_d concentrates all its mass on a facet of $\mathcal{S}_d$. In the latter case, however, $Y_j = 0$ almost surely in view of (2.3).

Because the random vector $\boldsymbol{Y} = (Y_1, \ldots, Y_d)$ in Proposition 4.1 has Fréchet margins, its unique copula is given, for all reals $u_1, \ldots, u_d \in (0, 1)$, by

$$\begin{aligned} C(u_1, \ldots, u_d) &= \Pr\left\{Y_1 \le F_1^{-1}(u_1), \ldots, Y_d \le F_d^{-1}(u_d)\right\} \\ &= \exp\left[-\mathrm{E}\left[\max_{j\in\{1,\ldots,d\}}\left\{|\ln(u_j)|\, Q_j^{\theta/\varrho}/\mathrm{E}(Q_j^{\theta/\varrho})\right\}\right]\right], \end{aligned}$$

owing to the fact that for every integer $j \in \{1, \ldots, d\}$, the quantile function of Y_j is given, for every real $u \in (0, 1)$, by $F_j^{-1}(u) = b_j|\ln(u)|^{-1/\theta}$. A simple calculation shows that the stable tail dependence function of C can be written, for all reals $x_1, \ldots, x_d \in [0, \infty)$, as

$$\ell(x_1, \ldots, x_d) = \mathrm{E}\left[\max_{j\in\{1,\ldots,d\}}\left\{x_j\, Q_j^{\theta/\varrho}/\mathrm{E}(Q_j^{\theta/\varrho})\right\}\right] = \mathrm{E}\left\{\max_{j\in\{1,\ldots,d\}}(x_j Z_j)\right\}, \tag{4.1}$$

where, for each integer $j \in \{1, \ldots, d\}$, $Z_j = Q_j^{\theta/\varrho}/\{\mathrm{E}(Q_j^{\theta/\varrho})\}$, which is non-negative and such that $\mathrm{E}(Z_j) = 1$.

The random vector $\boldsymbol{Z} = (Z_1, \ldots, Z_d)$ is the generator of the so-called D-norm defined, for every vector $\boldsymbol{y} = (y_1, \ldots, y_d) \in \mathbb{R}^d$, by $\|\boldsymbol{y}\|_{\boldsymbol{Z}} = \mathrm{E}\{\max_{j\in\{1,\ldots,d\}}(|y_i|\, Z_i)\}$, as described in Lemma 1.1.3 of [28]. Clearly, one then has $\ell(\boldsymbol{x}) = \|\boldsymbol{x}\|_{\boldsymbol{Z}}$ for all vectors $\boldsymbol{x} \in [0, \infty)^d$. However, notice that the distribution of the random vector $\boldsymbol{Z}/d$ is not necessarily supported on $\mathcal{S}_d$, unless $\theta = \varrho = 1$ and $\mathrm{E}(Q_1) = \cdots = \mathrm{E}(Q_d) = 1/d$. Consequently, the distribution of $\boldsymbol{Z}/d$ may not be the spectral measure of ℓ. However, Theorem 1.7.1 and Corollary 1.7.2 in [28] imply that there exists a uniquely determined spectral measure ς_d such that for any random vector $\boldsymbol{W}$ distributed as ς_d and all vectors $\boldsymbol{y} \in \mathbb{R}^d$, one has $\|\boldsymbol{y}\|_{\boldsymbol{Z}} = \|\boldsymbol{y}\|_{d\times\boldsymbol{W}}$. The exact form of ς_d is computed next.

Proposition 4.2 For every Borel subset B of $\mathcal{S}_d$, one has

$$\varsigma_d(B) = \sum_{i=1}^{d} \frac{1}{d\,\mathrm{E}(Q_i^{\theta/\varrho})} \int_{\psi^{-1}(B)} s_i^{\theta/\varrho}\,\mathrm{d}\sigma_d(s_1, \ldots, s_d),$$

where the map ψ: $[0,\infty)^d \setminus \{\mathbf{0}\} \to \mathcal{S}_d$ is given, for all reals $s_1, \ldots, s_d \in [0,\infty)$, by

$$\psi(s_1, \ldots, s_d) = \frac{\left(s_1{}^{\theta/\varrho}/\mathrm{E}(Q_1{}^{\theta/\varrho}), \ldots, s_d{}^{\theta/\varrho}/\mathrm{E}(Q_d{}^{\theta/\varrho})\right)}{\sum_{j=1}^{d} s_j{}^{\theta/\varrho}/\mathrm{E}(Q_j{}^{\theta/\varrho})}.$$

Proposition 4.1 invites the question of whether one can characterize the distribution of multivariate extreme-value vectors $\boldsymbol{Y}$ that belong to the class $\mathcal{M}_\varrho(\sigma_d)$ for some scalar $\varrho \in (0,\infty)$ and some probability measure σ_d on $\mathcal{S}_d$.

Before addressing this issue, recall again that if a random vector $\boldsymbol{Y}$ in the class $\mathcal{M}_\varrho(\sigma_d)$ has a multivariate extreme-value distribution, its margins are not degenerate, and hence in view of Proposition 4.1, the corresponding probability measure σ_d cannot concentrate all its mass on some facet of $\mathcal{S}_d$. In fact, if a random vector $\boldsymbol{Y} \in \mathcal{M}_\varrho(\sigma_d)$ has a multivariate extreme-value distribution, then its univariate margins must be Fréchet, given that any element of $\mathcal{M}_\varrho(\sigma_d)$ concentrates all its mass on $[0,\infty)^d$ by design. As shown next, the shape parameters of these univariate Fréchet distributions must also coincide.

Proposition 4.3 Let $\boldsymbol{Q} = (Q_1, \ldots, Q_d)$ be a random vector with distribution σ_d such that $Q_1 \not\equiv 0, \ldots, Q_d \not\equiv 0$ almost surely. Suppose that $\boldsymbol{Y} = (Y_1, \ldots, Y_d) \in \mathcal{M}_\varrho(\sigma_d)$ is multivariate extreme-value. Then, for every integer $j \in \{1, \ldots, d\}$, Y_j has a Fréchet distribution with location parameter $a_j = 0$, scale parameter $b_j \in (0,\infty)$, and shape parameter $\theta \in (0,\infty)$ which is common to all the components of $\boldsymbol{Y}$.

The next result follows from the characterization of multivariate extreme-value distributions with unit Fréchet margins (Section 5.4.1, [12]) and ideas from Section 1.7 of [28].

Proposition 4.4 Suppose that a random vector $\boldsymbol{Y} = (Y_1, \ldots, Y_d)$ has a multivariate extreme-value distribution and that, for every integer $j \in \{1, \ldots, d\}$, Y_j is Fréchet with location parameter $a_j = 0$, scale parameter $b_j \in (0,\infty)$, and shape parameter $\theta \in (0,\infty)$. Let $\gamma = b_1^\theta + \cdots + b_d^\theta$. Then the random vector $\boldsymbol{Y}$ belongs to the class $\mathcal{M}_\theta(\sigma_d)$, where:

(i) the probability measure σ_d on $\mathcal{S}_d$ is given, for every Borel subset A of $\mathcal{S}_d$, by $\sigma_d(A) = \mathrm{E}\{\|\boldsymbol{Q}^*\|_1 \times \mathbf{1}_A(\boldsymbol{Q}^*/\|\boldsymbol{Q}^*\|_1)\}$, where for each integer $j \in \{1, \ldots, d\}$, $Q_j^* = W_j b_j^\theta d/\gamma$ and the random vector $\boldsymbol{W} = (W_1, \ldots, W_d)$ is distributed as the spectral measure ς_d of $\boldsymbol{Y}$;

(ii) the radial measure of $\boldsymbol{Y}$ is given, for every real $t \in (0,\infty)$, by $S_\nu(t) = \gamma\, t^{-\theta}$.

In the case in which $\theta = b_1 = \cdots = b_d = 1$, Proposition 4.4 reduces to the well-known property of multivariate extreme-value distributions with unit Fréchet margins, namely that $\boldsymbol{Y} \in \mathcal{M}_1(\varsigma_d)$ with $S_\nu(t) = d/t$ for every real $t \in (0,\infty)$, as explained, for example, in Section 5.4.1 of [12]. It implies in particular that any extreme-value copula with spectral measure σ_d is the copula of some random vector belonging to the class $\mathcal{M}_1(\sigma_d)$.

To summarize, Propositions 4.1, 4.3, and 4.4 jointly lead to the following result.

Theorem 4.5 Fix a scalar $\varrho \in (0,\infty)$ and let $\boldsymbol{Q} = (Q_1, \ldots, Q_d)$ be a random vector with probability measure σ_d on $\mathcal{S}_d$ such that $Q_1 \not\equiv 0, \ldots, Q_d \not\equiv 0$ almost surely. Then a random vector $\boldsymbol{Y}$ in the class $\mathcal{M}_\varrho(\sigma_d)$ has a multivariate extreme-value distribution if and only if there exist scalars γ, $\theta \in (0,\infty)$ such that the generalized survival function S_ν associated with its radial measure ν satisfies $S_\nu(t) = \gamma\, t^{-\theta}$ for every real $t \in (0,\infty)$.

5. The case in which S_ν is regularly varying

Interestingly, the multivariate extreme-value distributions in Proposition 4.1 with spectral measures described in Proposition 4.2 arise as limiting distributions of maxima of vectors in $\mathcal{M}_\varrho(\sigma_d)$, provided that the survival function of their radial measure is regularly varying. To see this, recall that a measurable function $f\colon (0,\infty) \to [0,\infty)$ is regularly varying with index $\zeta \in \mathbb{R}$, if for every real $x \in (0,\infty)$, one has $\lim_{t\to\infty} f(tx)/f(t) = x^\zeta$. Moreover, a random vector $\boldsymbol{Y}$ with distribution function H is said to be in the maximum domain of attraction of a random vector $\boldsymbol{Y}^*$ with distribution function H^* if there exist sequences of vectors $\boldsymbol{a}_n \in \mathbb{R}^d$ and $\boldsymbol{b}_n \in (0,\infty)^d$ such that, for all vectors $\boldsymbol{x} \in \mathbb{R}^d$, one has $\lim_{n\to\infty} H^n(\boldsymbol{b}_n\boldsymbol{x} + \boldsymbol{a}_n) = H^*(\boldsymbol{x})$.

Proposition 5.1 Consider a random vector $\boldsymbol{Y}$ in the class $\mathcal{M}_\varrho(\sigma_d)$ for some scalar $\varrho \in (0,\infty)$ and a probability measure σ_d on $\mathcal{S}_d$ such that if $\boldsymbol{Q} = (Q_1,\ldots,Q_d)$ is a random vector with distribution σ_d, then $Q_1 \not\equiv 0,\ldots,Q_d \not\equiv 0$ almost surely. Suppose that the generalized survival function S_ν of the radial measure of $\boldsymbol{Y}$ is regularly varying with index $-\theta$ for some scalar $\theta \in (0,\infty)$. Then $\boldsymbol{Y}$ is in the maximum domain of attraction of a random vector $\boldsymbol{Y}^* \in \mathcal{M}_\varrho(\sigma_d)$ with radial measure ν^*, where for every real $t \in (0,\infty)$, $S_{\nu^*}(t) = \gamma\, t^{-\theta}$ with $\gamma = 1/\mathrm{E}\{\max({Q_1}^{\theta/\varrho},\ldots,{Q_d}^{\theta/\varrho})\}$.

As an illustration, consider the case treated in Section 3 in which σ_d is Dirichlet.

Example 5.2 Let $\boldsymbol{Q} = (Q_1,\ldots,Q_d)$ be a random vector with Dirichlet distribution $\sigma_d = \Delta_{\boldsymbol{\alpha}}$. Then, for every integer $j \in \{1,\ldots,d\}$, Q_j is distributed as $\mathcal{B}(\alpha_j, \alpha_+ - \alpha_j)$ with $\alpha_+ = \alpha_1 + \cdots + \alpha_d$, and for any scalars θ, $\varrho \in (0,\infty)$, one has

$$\mathrm{E}(Q_j^{\theta/\varrho}) = \frac{\Gamma(\alpha_+)\Gamma(\alpha_j + \theta/\varrho)}{\Gamma(\alpha_j)\Gamma(\alpha_+ + \theta/\varrho)},$$

where $\Gamma(\cdot)$ is Euler's gamma function. Next suppose that $\boldsymbol{Y}^* \in \mathcal{M}_\varrho(\Delta_{\boldsymbol{\alpha}})$ with radial measure ν^* given, for some scalar $\gamma \in (0,\infty)$ and every real $t \in (0,\infty)$, by $S_{\nu^*}(t) = \gamma\, t^{-\theta}$. Proposition 4.1 then implies that $\boldsymbol{Y}^*$ has a multivariate extreme-value distribution; its stable tail dependence function is then given, for all reals $x_1,\ldots,x_d \in [0,\infty)$, by

$$\ell(x_1,\ldots,x_d) = \frac{\Gamma(\alpha_+ + \theta/\varrho)}{\Gamma(\alpha_+)}\,\mathrm{E}\left[\max_{j\in\{1,\ldots,d\}}\left\{\frac{x_j Q_j^{\theta/\varrho}\Gamma(\alpha_j)}{\Gamma(\alpha_j + \theta/\varrho)}\right\}\right].$$

This map can be recognized as the positive scaled extremal Dirichlet stable tail dependence function with parameter $\rho = \theta/\varrho$ and $\boldsymbol{\alpha}$ introduced in Definition 1 of [29]. As discussed on p. 74 in [29], when $\boldsymbol{\alpha} = \mathbf{1}$, the above map ℓ reduces to the stable tail dependence function of the Galambos copula with parameter θ/ϱ, as defined in [30]. When $\theta = \varrho$, ℓ is the stable tail dependence function of the extremal Dirichlet model of [31].

When the univariate margins of a random vector $\boldsymbol{Y}$ in $\mathcal{M}_\varrho(\sigma_d)$ are continuous, Proposition 5.1, in combination with Theorem 7.48 and Proposition 7.51 in [32] and (4.1), allows one to compute the upper-tail dependence coefficient of Joe [33], which is commonly used to summarize dependence in the upper tail of a bivariate distribution. It is defined, for all distinct integers $i,j \in \{1,\ldots,d\}$ by $\lambda_u(i,j) = \lim_{q\uparrow 1} \Pr\{Y_i > F_i^{-1}(q) \mid Y_j > F_j^{-1}(q)\}$, provided that the limit exists, where F_i and F_j are the distribution functions of Y_i and Y_j, respectively. This observation is summarized below.

Corollary 5.3 Consider a random vector $\boldsymbol{Y}$ from the class $\mathcal{M}_\varrho(\sigma_d)$ for some scalar $\varrho \in (0,\infty)$ and a probability measure σ_d on $\mathcal{S}_d$. Suppose that the univariate margins of $\boldsymbol{Y}$

are continuous and that the generalized survival function S_ν of the radial measure of $\boldsymbol{Y}$ is regularly varying with index $-\theta$ for some scalar $\theta \in (0, \infty)$. Then for any distinct integers $i, j \in \{1, \ldots, d\}$,

$$\lambda_u(i,j) = 2 - \mathrm{E}\left[\max\left\{Q_i^{\theta/\varrho}/\mathrm{E}(Q_i^{\theta/\varrho}), Q_j^{\theta/\varrho}/\mathrm{E}(Q_j^{\theta/\varrho})\right\}\right].$$

Going beyond a one-number summary such as the upper-tail dependence coefficient, Theorem 7.48 in [32] guarantees that, under the conditions of Proposition 5.1, the copula of a random vector $\boldsymbol{Y}$ with continuous univariate margins is in the domain of attraction of the unique (extreme-value) copula of $\boldsymbol{Y}^*$. Recall that a copula C is in the maximum domain of attraction of another copula C^* if, for all vectors $\boldsymbol{u} \in [0,1]^d$, $\lim_{n\to\infty} C^n(\boldsymbol{u}^{1/n}) = C^*(\boldsymbol{u})$. Notably, from Example 5.2, one can easily deduce the extremal behavior of reciprocal Liouville copulas, as summarized below.

Corollary 5.4 Consider a reciprocal Liouville copula $C^\nu_{1,\boldsymbol{\alpha}}$ with the property that its corresponding generalized survival function S_ν is regularly varying with index $-\theta$ for some scalar $\theta \in (0, \infty)$. Then $C^\nu_{1,\boldsymbol{\alpha}}$ belongs to the domain of attraction of the extreme-value copula C^* with positive scaled Dirichlet stable tail dependence function with parameters θ and $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_d) \in (0,\infty)^d$. When $\alpha_1 = \cdots = \alpha_d = 1$, C^* is the Galambos copula with parameter θ.

As mentioned earlier, the copula $C^\nu_{1,\boldsymbol{\alpha}}$ is reciprocal Archimedean when $\alpha_1 = \cdots = \alpha_d = 1$. The fact that it is attracted to the Galambos copula was also derived in [34], albeit under a different condition, namely that its generator F is such that $1 - F$ is regularly varying with index $-\theta$ for some scalar $\theta \in (0, \infty)$.

Remark 5.5 In view of the above discussion, the parallel between the Gumbel and Galambos copulas drawn in [34] can be generalized to extreme-value copulas with scaled Dirichlet stable tail dependence functions: under suitable conditions, the stable tail dependence functions of the attractors of Liouville copulas are negative scaled extremal Dirichlet (Corollary 1 in [29]), while those of reciprocal Liouville copulas are positive scaled extremal Dirichlet (Corollary 5.4).

6. Stochastic representation and simulation

Analogous to Proposition 4 in [9], which concerns $\mathcal{M}_1(\upsilon_d)$, a stochastic representation also exists for elements of $\mathcal{M}_\varrho(\sigma_d)$. In what follows, δ_t denotes the Dirac measure at $t \in \mathbb{R}$.

Proposition 6.1 Let $\boldsymbol{Y}$ be a random vector from the class $\mathcal{M}_\varrho(\sigma_d)$ with exponent measure μ and radial measure ν. Further let

(i) $\boldsymbol{Q}_1, \boldsymbol{Q}_2, \ldots$ be a sequence of mutually independent and identically distributed random variables, each of whose terms is distributed as σ_d;
(ii) $\zeta_\nu = \delta_{R_1} + \cdots + \delta_{R_N}$ be a Poisson point process on $(0, \infty]$ with mean measure ν which is independent of $\boldsymbol{Q}_1, \boldsymbol{Q}_2, \ldots$

In statement (ii) above, $N = \infty$ unless $\nu(0,\infty] = u_\nu < \infty$, in which case N is a Poisson random variable with mean u_ν. Then $\boldsymbol{Y}$ is distributed as $\max(\boldsymbol{0}, R_1\boldsymbol{Q}_1{}^{1/\varrho}, \ldots, R_N\boldsymbol{Q}_N{}^{1/\varrho})$, where the maximum is interpreted as $\boldsymbol{0}$ if $N = 0$.

When $\nu(0,\infty] = u_\nu < \infty$, simulation of a random vector $\boldsymbol{Y}$ in $\mathcal{M}_\varrho(\sigma_d)$ is a straightforward application of the well-known construction of Poisson point processes with finite intensity. For completeness, it is reproduced here as Algorithm A. Therein, S_ν^{-1} denotes

Algorithm A

For any given scalar $\varrho \in (0,\infty)$, probability measure σ_d on the simplex $\mathcal{S}_d$, and radial measure ν with $u_\nu < \infty$, let $\boldsymbol{Y}$ be a random vector from the class $\mathcal{M}_\varrho(\sigma_d)$ with radial measure ν and associated generalized survival function S_ν. To generate an observation from $\boldsymbol{Y}$, proceed as follows:

Require: $\varrho > 0$, $u_\nu \in (0,\infty)$
1: **sample** $N \sim \mathcal{P}(u_\nu)$ ▷ Poisson with mean u_ν
2: **if** $N = 0$ **then**
3: $\quad \boldsymbol{Y} \leftarrow \boldsymbol{0}$
4: **else**
5: $\quad$ **for all** $i \in \{1,\dots,N\}$ **do**
6: $\quad\quad$ **sample** $U_i \sim \mathcal{U}(0,1)$ ▷ Uniform on $(0,1)$
7: $\quad\quad R_i \leftarrow S_\nu^{-1}(u_\nu U_i)$
8: $\quad\quad$ **sample** independently $\boldsymbol{Q}_i$ from distribution σ_d
9: $\quad$ **end for**
10: $\quad \boldsymbol{Y} \leftarrow \max(R_1 \boldsymbol{Q}_1^{1/\varrho}, \dots, R_N \boldsymbol{Q}_N^{1/\varrho})$ ▷ Component-wise power and max
11: **end if**
12: **return** $\boldsymbol{Y}$

Algorithm B

For any given scalar $\varrho \in (0,\infty)$, probability measure σ_d on the simplex $\mathcal{S}_d$, and radial measure ν with $u_\nu = \infty$, let $\boldsymbol{Y}$ be a random vector from the class $\mathcal{M}_\varrho(\sigma_d)$ with radial measure ν and associated generalized survival function S_ν. To generate an observation from $\boldsymbol{Y}$, proceed as follows:

Require: $\varrho > 0$
1: **initialize** $\boldsymbol{Y} \leftarrow \boldsymbol{0}$
2: **sample** $\varepsilon \sim \mathcal{E}(1)$ ▷ Unit exponential
3: **assign** $T \leftarrow \varepsilon$ and $R \leftarrow S_\nu^{-1}(T)$
4: **while** $R > \min(\boldsymbol{Y})$ **do**
5: $\quad$ **sample** $\boldsymbol{Q}$ from distribution σ_d
6: $\quad \boldsymbol{Y} \leftarrow \max(\boldsymbol{Y}, R\boldsymbol{Q}^{1/\varrho})$ ▷ Component-wise power and max
7: $\quad$ **sample** $\varepsilon \sim \mathcal{E}(1)$
8: $\quad T \leftarrow T + \varepsilon$ and $R \leftarrow S_\nu^{-1}(T)$
9: **end while**
10: **return** $\boldsymbol{Y}$

the pseudo-inverse of S_ν defined as $S_\nu^{-1}(s) = \inf\{t > 0\colon S_\nu(t) \le s\}$ for all $s \in [0, u_\nu]$.

The case $u_\nu = \infty$ is more challenging. Fortunately, one can use the algorithm designed in [10], which is again provided here as Algorithm B for completeness.

The main numerical challenges in implementing Algorithms A and B are the evaluation of the pseudo-inverse S_ν^{-1} and the generation of observations $\boldsymbol{Q}$ from distribution σ_d. A particularly tractable case is that of the Dirichlet distributions $\Delta_{\boldsymbol{\alpha}}$ with parameter $\boldsymbol{\alpha} \in (0,\infty)^d$ already discussed in Section 3.

Drawing an observation $\boldsymbol{Q}$ from $\Delta_{\boldsymbol{\alpha}}$ in Step 5 of Algorithm B is straightforward. It suffices to set $Q_j = W_j/(W_1 + \cdots + W_d)$ for every integer $j \in \{1,\dots,d\}$, where $W_1,\dots,W_d$ are mutually independent random variables such that W_j is Gamma with shape parameter α_j and scale parameter 1. Likewise, the inverse S_ν^{-1} is easily calculated when $S_\nu(t) = \gamma t^{-\theta}$ for all $t \in (0,\infty)$ and some $\gamma, \theta \in (0,\infty)$. This special case is detailed in Algorithm C.

Algorithm C

For any given scalars $\gamma, \theta \in (0, \infty)$ and vector $\boldsymbol{\alpha} \in (0, \infty)^d$, let $\boldsymbol{Y}$ be a random vector from the class $\mathcal{M}_\varrho(\Delta_{\boldsymbol{\alpha}})$ with generalized survival defined, for every real $t \in (0, \infty)$, by $S_\nu(t) = \gamma t^{-\theta}$. To generate an observation from $\boldsymbol{Y}$, proceed as follows:

Require: $\varrho > 0,\ \theta > 0,\ \gamma > 0,\ \boldsymbol{\alpha} \in (0, \infty)^d$

initialize $\boldsymbol{Y} \leftarrow \mathbf{0}$
sample $\varepsilon \sim \mathcal{E}(1)$ ▷ Unit exponential
assign $T \leftarrow \varepsilon$ and $R \leftarrow (\gamma/T)^{1/\theta}$
while $R > \min(\boldsymbol{Y})$ **do**
  for all $j \in \{1, \ldots, d\}$ **do**
    sample independently $W_j \sim \mathcal{G}(\alpha_j, 1)$ ▷ Gamma with shape α_j and scale 1
  end for
  $\boldsymbol{Q} \leftarrow \boldsymbol{W}/(W_1 + \cdots + W_d)$
  $\boldsymbol{Y} \leftarrow \max\left(\boldsymbol{Y},\ R\boldsymbol{Q}^{1/\varrho}\right)$ ▷ Component-wise power and max
  sample $\varepsilon \sim \mathcal{E}(1)$
  $T \leftarrow T + \varepsilon$ and $R \leftarrow (\gamma/T)^{1/\theta}$
end while
return $\boldsymbol{Y}$

Example 6.2 The left plot in Figure 1 displays random samples of size 1000 from the bivariate reciprocal Liouville copula $C^{\nu}_{1,\boldsymbol{\alpha}}$ with parameter $\boldsymbol{\alpha} = (\alpha_1, \alpha_2) = (1, 10)$ and radial measure ν with generalized survival function S_ν parametrized by a scalar $\theta \in (0, \infty)$ and given, for every real $t \in (0, \infty)$, by

$$S_\nu(t) = \frac{\theta\,\Gamma(d + 1/\theta)}{\Gamma(d)\Gamma(1/\theta)}\, t^{-1/\theta}. \tag{6.1}$$

In this illustration, $\theta = 1$. The right plot in Figure 1 corresponds to the same $\boldsymbol{\alpha}$ and θ, but takes the radial measure to be $\nu_{\varrho,1}$ with $\varrho = 0.5$. In this case, $C^{\nu_{\varrho,1}}_{1,\boldsymbol{\alpha}}$ is also the copula of the max-id distribution in $\mathcal{M}_\varrho(\Delta_\alpha)$ with radial measure ν. Since $\alpha_1 \neq \alpha_2$, the reciprocal Liouville copulas are non-exchangeable, a feature clearly visible in the two graphs.

Example 6.3 As already mentioned, the caveat of Algorithm B is the inversion of S_ν, if the latter is not tractable. When $\sigma_d = \upsilon_d$, this can occur even when the generator Λ of the

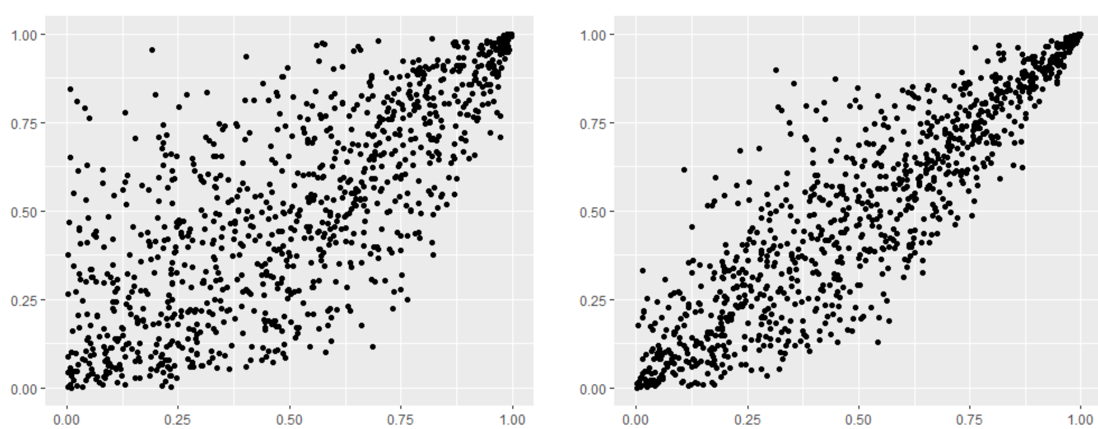

Figure 1. Scatter plots of random samples of size 1000 from a max-id random vector in $\mathcal{M}_\varrho(\sigma_d)$ in dimension $d = 2$ when σ_d is Dirichlet on $\mathcal{S}_d$ with parameters $\alpha_1 = 1$ and $\alpha_2 = 10$ and the generalized survival function S_ν associated with the exponent measure μ is of the form (6.1) with $\theta = 1$. The graphs, in which $\varrho = 1$ (left) and $\varrho = 1/2$ (right), are displayed on the quantile scale to remove the effect of the marginals.

max-id distribution with ℓ_ϱ-quasinorm symmetric exponent measure and radial measure ν is in closed form. Indeed, from Theorem 3.6, $S_\nu(t) = \mathfrak{W}_d^{-1}(\Lambda)(t^\varrho)$ for every real $t \in (0,\infty)$, where the inverse Williamson d-transform of Λ involves its higher-order derivatives.

The work of Mai and Wang [11] offers an elegant way to embed the generator Λ with tractable S_ν into a parametric class $\{\Lambda_p\colon p \geq 1\}$ of outer power transforms while preserving the feasibility of the random number generation mechanism.

Specifically, let $\Lambda_p(t) = \Lambda(t^{1/p})$ for every real $t \in (0,\infty)$, and suppose that $\Lambda(t) \to \infty$ as $t \to 0$. Proposition 3 given in [23] then shows that Λ_p is a valid generator of a random vector $\boldsymbol{Y}$ with ℓ_ϱ-quasinorm symmetric exponent measure. It is easily seen from the proof of Lemma 7 in [11] that if the random vector $\boldsymbol{Z}$ is sampled through Algorithm 2 of Mai and Wang [11] with $G_\nu = S_\nu$, an observation from $\boldsymbol{Y}$ is given by $\boldsymbol{Z}^{p/\varrho}$.

Remark 6.4 The stochastic representation in Proposition 6.1 allows one to see how the class $\mathcal{M}_1(\sigma_d)$ relates to one other class of max-id distributions proposed in the literature, namely the generalized spectral construction in Section 3.3 of [6]. In the latter article, the finite-dimensional distributions of the max-id process have a similar stochastic representation, except that $\boldsymbol{Q}$ is replaced by a non-negative random vector $\boldsymbol{W}$ with finite, non-zero marginal expectations. Mimicking the argument in Proposition 3.1, one can deduce that the construction in [6] corresponds to an element of $\mathcal{M}_1(\sigma_d)$ as soon as $\boldsymbol{W} = (W_1,\ldots,W_d)$ has the property that $W_1 + \cdots + W_d$ is independent of $\{1/(W_1 + \cdots + W_d)\}\boldsymbol{W}$. Of course, this is trivially true if $\boldsymbol{W}$ is supported on $\mathcal{S}_d$. However, it is not true when $\boldsymbol{W}$ is (truncated) Gaussian, which is the focus of [6].

7. Conclusions

This article studied a new broad class of max-id distributions and some of its properties. In particular, the proposed distributions are non-exchangeable as soon as the measure σ_d on the unit simplex has this property; they are also asymptotically dependent in the upper tail whenever the survival function of the radial measure ν is regularly varying.

Two extensions suggest themselves for future work. First, extending the present construction to max-id stochastic processes, as in [6], would be of interest. Second, exploring the practical applications of the class $\mathcal{M}_\varrho(\sigma_d)$ appears worthwhile. This would require developing statistical inference methods, including the selection of ν and σ_d, followed by model estimation and validation. Although these tasks lie beyond the scope of this article, a promising estimation strategy could involve using the pairwise composite likelihood.

Appendix

Proof of Lemma 2.4

To prove statement (i), fix an arbitrary real $x \in (0,\infty)$. Given that the set $A_x = \{\boldsymbol{x} \in E_d\colon \|\boldsymbol{x}\|_\varrho > x\}$ is bounded away from $\mathbf{0}$, $\nu(x,\infty] = \mu(A_x) < \infty$ because μ is a Radon measure. Furthermore, $\cap_{x>0} A_x = \cup_{j=1}^d \{\boldsymbol{y} \in E_d\colon y_j = \infty\}$ and hence $\nu\{\infty\} = \mu(\bigcap_{x>0} A_x) = 0$ by the property (1.2) of μ.

Turning to statement (ii), fix a vector $\boldsymbol{x} \in E_d$ and let $[-\boldsymbol{\infty}, \boldsymbol{x}]^\complement$ be the complement of $[-\boldsymbol{\infty}, \boldsymbol{x}]$ in E_d. If $\boldsymbol{Q} = (Q_1,\ldots,Q_d)$ is a random vector with distribution σ_d, then

$$\begin{aligned}\mu[-\boldsymbol{\infty}, \boldsymbol{x}]^\complement &= \nu \otimes \sigma_d\big\{(r,\boldsymbol{s}) \in (0,\infty] \times \mathcal{S}_d\colon \exists_{j\in\{1,\ldots,d\}}\ r s_j^{1/\varrho} > x_j\big\} \\ &= \int_{\mathcal{S}_d} \nu\big\{r \in (0,\infty]\colon r > \min(\boldsymbol{x}/\boldsymbol{s}^{1/\varrho})\big\}\, \mathrm{d}\sigma_d(\boldsymbol{s}),\end{aligned}$$

and hence

$$\mu[-\boldsymbol{\infty},\boldsymbol{x}]^{\complement} = \int_{\mathcal{S}_d} S_\nu\big\{\min(\boldsymbol{x}/\boldsymbol{s}^{1/\varrho})\big\}\,\mathrm{d}\sigma_d(\boldsymbol{s}) = \mathrm{E}\big[S_\nu\big\{\min(\boldsymbol{x}/\boldsymbol{Q}^{1/\varrho})\big\}\big]. \tag{A.1}$$

Because S_ν is non-increasing, $\mu[-\boldsymbol{\infty},\boldsymbol{x}]^{\complement} < S_\nu\{\min(\boldsymbol{x})\} < \infty$ for all $\boldsymbol{x} \in (0,\infty)^d$. Also, $\mu[-\boldsymbol{\infty},\boldsymbol{x}]^{\complement} \to 0$ as $\boldsymbol{x} \to \boldsymbol{\infty}$. Thus, by Lemma 4 in [9], μ is an exponent measure on E_d.

Proof of Proposition 2.5

Proposition 5.8 in [12] implies that $\Pr(\boldsymbol{Y} \le \boldsymbol{y}) = \exp\{-\mu(-\boldsymbol{\infty},\boldsymbol{y})^{\complement}\}$ whenever $\boldsymbol{y} \in [0,\infty)^d$ and equals zero otherwise. Therefore, the first formula follows at once from identity (A.1) in the proof of Lemma 2.4. The second follows from the fact that, for any $\boldsymbol{y} \in E_d$,

$$\begin{aligned}\nu \otimes \sigma_d\Big\{(r,\mathbf{s}) \in (0,\infty] \times \mathcal{S}_d\colon \exists_{j\in\{1,\ldots,d\}}\; r s_j^{1/\varrho} > y_j\Big\}\\ = \int_0^\infty \sigma_d\Big\{\mathbf{s} \in \mathcal{S}_d\colon \exists_{j\in\{1,\ldots,d\}}\; s_j > y_j^{\varrho}/r^{\varrho}\Big\}\,\mathrm{d}\nu(r).\end{aligned}$$

whence $\mu[-\boldsymbol{\infty},\boldsymbol{y}]^{\complement} = \int_0^\infty \{\mathbf{1} - H(\boldsymbol{y}^{\varrho}/r^{\varrho})\}\,\mathrm{d}\nu(r)$.

As for the third expression, it is a straightforward consequence of identity (B.3) in [9] and the fact that for any vector $\boldsymbol{x} \in E_d$, one has

$$\begin{aligned}\mu(\boldsymbol{x},\boldsymbol{\infty}] &= \nu \otimes \sigma_d\big\{(r,\boldsymbol{s}) \in (0,\infty] \times \mathcal{S}_d\colon \forall_{j\in\{1,\ldots,d\}}\; r s_j^{1/\varrho} > x_j\big\}\\ &= \int_0^\infty \bar{H}(\boldsymbol{x}^{\varrho}/r^{\varrho})\,d\nu(r) = \int_{\max(\boldsymbol{x})}^\infty \bar{H}(\boldsymbol{x}^{\varrho}/r^{\varrho})\,\mathrm{d}\nu(r),\end{aligned} \tag{A.2}$$

where the last step is justified by the fact that $\bar{H}(\boldsymbol{t}) = 0$ whenever at least one of the components of $\boldsymbol{t} = (t_1,\ldots,t_d)$ is greater than 1, that is, $t_j \in (1,\infty)$ for at least one $j \in \{1,\ldots,d\}$.

Proof of Lemma 2.6

Fix an arbitrary vector $\boldsymbol{y} \in [0,\infty)^d$. From the first expression in Proposition 2.5, one has

$$\begin{aligned}\Pr(\boldsymbol{Z}^{\varrho^*/\varrho} \le \boldsymbol{y}) = \Pr(\boldsymbol{Z} \le \boldsymbol{y}^{\varrho/\varrho^*}) &= \exp\Big[-\mathrm{E}\big[S_{\nu_{\varrho,\varrho^*}}\{\min(\boldsymbol{y}^{\varrho/\varrho^*}/\boldsymbol{Q}^{1/\varrho^*})\}\big]\Big]\\ &= \exp\Big[-\mathrm{E}\big[S_\nu[\{\min(\boldsymbol{y}^{\varrho/\varrho^*}/\boldsymbol{Q}^{1/\varrho^*})\}^{\varrho^*/\varrho}]\big]\Big]\\ &= \exp\Big[-\mathrm{E}\big[S_\nu\{\min(\boldsymbol{y}/\boldsymbol{Q}^{1/\varrho})\}\big]\Big] = \Pr(\boldsymbol{Y} \le \boldsymbol{y}).\end{aligned}$$

The random vectors $\boldsymbol{Y}$ and $\boldsymbol{Z}^{\varrho^*/\varrho}$ thus have the same distribution. If the latter is continuous, they also share the same unique copula. Moreover, given that the map $x \mapsto x^{\varrho^*/\varrho}$ is strictly increasing on $[0,\infty)$, the copulas of $\boldsymbol{Z}^{\varrho^*/\varrho}$ and $\boldsymbol{Z}$ coincide.

Proof of Proposition 2.7

First note that statement (i) is a special case of statement (ii) when $K = d$ and $A_k = \{k\}$ for every integer $k \in \{1,\ldots,d\}$ because σ_{A_k} is then degenerate, placing all its mass at

$\boldsymbol{e}_k$. To prove statement (ii), let $\boldsymbol{Q}_{A_k}$ be a random vector distributed as σ_{A_k}. For each vector $\boldsymbol{y} \in [0,\infty)^d$, let also $\boldsymbol{y}_{A_k}$ be the subvector $(y_j\colon j \in A_k)$ and $\boldsymbol{y}^*_{A_k}$ be the vector with components $y^*_{A_k,j} = y_j$ if $j \in A_k$ and $y^*_{A_k,j} = \infty$ if $j \notin A_k$.

Then, it follows from Proposition 2.5 that, for each integer $k \in \{1,\ldots,K\}$, one has

$$\begin{aligned}\Pr(\boldsymbol{Y}_{A_k} \le \boldsymbol{y}_{A_k}) = \Pr(\boldsymbol{Y} \le \boldsymbol{y}^*_{A_k}) &= \exp\left[-\sum_{\ell=1}^K p_\ell \mathrm{E}\Big[S_\nu\{\min(\boldsymbol{y}^*_{A_k}/\boldsymbol{Q}_{A_\ell}{}^{1/\varrho})\}\Big]\right]\\ &= \exp\Big[-p_k \mathrm{E}\Big[S_\nu\Big\{\min_{j\in A_k}(y_j/Q_{A_k,j}{}^{1/\varrho})\Big\}\Big]\Big]\end{aligned}$$

because $S_\nu\{\min(\boldsymbol{y}^*_{A_k}/\boldsymbol{Q}_{A_\ell}{}^{1/\varrho})\} = 0$ almost surely if $k \neq \ell$, given that $Q_{A_\ell,j} = 0$ almost surely if $j \notin A_\ell$, and $S_\nu(\infty) = 0$.

Furthermore, one can similarly deduce that for every integer $k \in \{1,\ldots,d\}$, one has $S_\nu\{\min(\boldsymbol{y}/\boldsymbol{Q}_{A_k}{}^{1/\varrho})\} = S_\nu\{\min_{j\in A_k}(y_j/Q_{A_k,j}{}^{1/\varrho})\}$ almost surely. Consequently,

$$\begin{aligned}\Pr(\boldsymbol{Y} \le \boldsymbol{y}) &= \exp\left[-\sum_{k=1}^K p_k \mathrm{E}\big[S_\nu\{\min(\boldsymbol{y}/\boldsymbol{Q}_{A_k}{}^{1/\varrho})\}\big]\right]\\ &= \exp\left[-\sum_{k=1}^K p_k \mathrm{E}\Big[S_\nu\Big\{\min_{j\in A_k}(y_j/Q_{A_k,j}{}^{1/\varrho})\Big\}\Big]\right] = \prod_{k=1}^K \Pr(\boldsymbol{Y}_{A_k} \le \boldsymbol{y}_{A_k}).\end{aligned}$$

Turning to statement (iii), first deduce from (2.2) that, for every vector $\boldsymbol{y} \in [0,\infty)^d$ and integer $j \in \{1,\ldots,d\}$, one has $\Pr(Y_j \le y_j) = \exp\{-S_\nu(y/q_j^{1/\varrho})\}$. Furthermore, from Proposition 2.5 and the fact that S_ν is non-increasing, one has, for every vector $\boldsymbol{y} \in [0,\infty)^d$,

$$\begin{aligned}\Pr(\boldsymbol{Y} \le \boldsymbol{y}) &= \exp\Big[-S_\nu\Big\{\min_{j\in\{1,\ldots,d\}}(y_j/q_j^{1/\varrho})\Big\}\Big]\\ &= \min_{j\in\{1,\ldots,d\}}\big[\exp\{-S_\nu(y_j/q_j^{1/\varrho})\}\big] = \min_{j\in\{1,\ldots,d\}}\{\Pr(Y_j \le y_j)\}.\end{aligned}$$

Consequently, a possible copula of $\boldsymbol{Y}$ is the Fréchet–Hoeffding upper bound, defined for all reals $u_1,\ldots,u_d \in [0,1]$ by $M(u_1,\ldots,u_d) = \min(u_1,\ldots,u_d)$. In view of Theorem 2 in [35], the components of $\boldsymbol{Y}$ are thus comonotonic.

Proof of Proposition 3.1

First note that the support of $W_{\mathcal{I}}$ is necessarily contained in the interval $[0,1]$ because $\boldsymbol{Q}$ is supported on $\mathcal{S}_d$. Hence, for every real $x \in (0,\infty)$, one has

$$\nu^*_{\mathcal{I}}(x,\infty] = \int_0^1 \nu(x/w^{1/\varrho},\infty]\,\mathrm{d}F_{\mathcal{I}}(w) \le \int_0^1 \nu(x,\infty]\,\mathrm{d}F_{\mathcal{I}}(w) = \nu(x,\infty] < \infty.$$

Furthermore, $\nu^*_{\mathcal{I}}\{\infty\} = \lim_{x\to\infty}\nu^*_{\mathcal{I}}(x,\infty] \le \lim_{x\to\infty}\nu(x,\infty] = \nu\{\infty\} = 0$, which shows that $\nu^*_{\mathcal{I}}$ is a radial measure. From (2.1), one has, for every real $y_i \in [0,\infty)$ with $i \in \mathcal{I}$,

$$\Pr(\boldsymbol{Y}_{\mathcal{I}} \le \boldsymbol{y}_{\mathcal{I}}) = \exp\Big[-\mathrm{E}\big[S_\nu\{\min(\boldsymbol{y}_{\mathcal{I}}/\boldsymbol{Q}_{\mathcal{I}}^{1/\varrho})\}\big]\Big].$$

Using the independence of W and $\boldsymbol{Q}_{\mathcal{I}}^*$, the expectation $\mathrm{E}\big[S_\nu\{\min(\boldsymbol{y}_{\mathcal{I}}/\boldsymbol{Q}_{\mathcal{I}}{}^{1/\varrho})\}\big]$ on the right-hand side can be rewritten as

$$\mathrm{E}\left[\int_0^1 S_\nu\{\min(\boldsymbol{y}_{\mathcal{I}}/\boldsymbol{Q}_{\mathcal{I}}^{*\,1/\varrho})/w^{1/\varrho}\}\,\mathrm{d}F_{\mathcal{I}}(w)\right] = \mathrm{E}\big[S_{\nu_{\mathcal{I}}^*}\{\min(\boldsymbol{y}_{\mathcal{I}}/\boldsymbol{Q}_{\mathcal{I}}^{*\,1/\varrho})\}\big].$$

The claim then follows at once from Proposition 2.5.

Proof of Proposition 3.5

Suppose that $\Lambda\colon (0,\infty)\to[0,\infty)$ is the generator of an ℓ_ϱ-quasinorm symmetric exponent measure μ, say. Define a measure μ_ϱ on E_d through the relation $\mu_\varrho(A)=\mu\{\boldsymbol{y}\in E_d\colon \boldsymbol{y}^\varrho\in A\}$ for every Borel set $A\subseteq E_d$. Clearly, μ_ϱ is an exponent measure that satisfies condition (3.1). Furthermore, for every vector $\boldsymbol{x}\in E_d$, one has

$$\begin{aligned}\mu_\varrho(\boldsymbol{x},\boldsymbol{\infty}] &= \mu\big\{\boldsymbol{y}\in E_d\colon \forall_{j\in\{1,\dots,d\}}\ y_j^\varrho > x_j\big\}\\ &= \mu\big\{\boldsymbol{y}\in E_d\colon \forall_{j\in\{1,\dots,d\}}\ y_j > x_j^{1/\varrho}\big\} = \Lambda\big(\|\boldsymbol{x}^{1/\varrho}\|_\varrho^\varrho\big) = \Lambda(\|\boldsymbol{x}\|_1),\end{aligned}$$

showing that μ_ϱ is ℓ_1-norm symmetric with generator Λ. Proposition 2 in [9] implies that Λ is d-monotone on $(0,\infty)$ and vanishes at infinity. The same proposition also implies the converse: if Λ has the stated properties, it is then the generator of an ℓ_1-norm symmetric exponent measure, say μ^*. Retracing steps, one can readily see that the exponent measure $\mu^*_{1/\varrho}$ defined, for every Borel set $A\subseteq E_d$, by $\mu^*_{1/\varrho}(A)=\mu^*\big\{\boldsymbol{y}\in E_d\colon \boldsymbol{y}^{1/\varrho}\in A\big\}$, satisfies condition (3.1) and is such that, for every vector $\boldsymbol{x}\in E_d$,

$$\mu^*_{1/\varrho}(\boldsymbol{x},\boldsymbol{\infty}] = \mu^*(\boldsymbol{x}^\varrho,\boldsymbol{\infty}] = \Lambda(\|\boldsymbol{x}^\varrho\|_1) = \Lambda(\|\boldsymbol{x}\|_\varrho^\varrho).$$

In conclusion, $\mu^*_{1/\varrho}$ is an ℓ_q-quasinorm symmetric exponent measure with generator Λ.

Proof of Theorem 3.6

First suppose that the image of μ by the transformation $\mathcal{T}_\varrho$ is of the form $\nu\times\upsilon_d$ for some radial measure ν. From (A.2), one gets, for every vector $\boldsymbol{x}\in E_d$,

$$S_\mu(\boldsymbol{x}) = \int_0^\infty \big(1-\|\boldsymbol{x}\|_\varrho^\varrho/r^\varrho\big)_+^{d-1}\,\mathrm{d}\nu(r) = \int_0^\infty \big(1-\|\boldsymbol{x}\|_\varrho^\varrho/s\big)_+^{d-1}\,\mathrm{d}\nu_{\varrho,1}(s)$$

by the change of variable $r^\varrho\mapsto s$. Defining $\Lambda=\mathfrak{W}_d(\nu_{\varrho,1})$ as the Williamson d-transform of $\nu_{\varrho,1}$, one can write the right-most expression in the above display as $\Lambda(\|\boldsymbol{x}\|_\varrho^\varrho)$. Therefore, $S_\mu(\boldsymbol{x})=\Lambda(\|\boldsymbol{x}\|_\varrho^\varrho)$ for every vector $x\in E_d$, and $\Lambda(t)\to 0$ as $t\to\infty$. To show that condition (3.1) holds, one can mimic the calculation on top of p. 3784 in [9]. For any sequence $(\boldsymbol{z}_n)$ of vectors such that $\boldsymbol{z}_n\in(0,\infty)^d$ for every integer $n\in\mathbb{N}$ and $\boldsymbol{z}_n\to\boldsymbol{0}$ as $n\to\infty$, one has

$$\mu\big\{\boldsymbol{x}\in E_d\colon x_k=0\big\} = \lim_{n\to\infty}\mu\{\boldsymbol{x}\in E_d\colon x_k=0, \forall_{j\neq k}\ x_j > z_{jn}\},$$

where for each integer $n \in \mathbb{N}$,

$$\mu\{\boldsymbol{x} \in E_d\colon x_k = 0, \forall_{j\neq k}\, x_j > z_{jn}\}$$
$$= \nu \times \upsilon_d\big\{r \in (0,\infty], \boldsymbol{s} \in \mathcal{S}_d\colon r s_k^{1/\varrho} = 0, \forall_{j\neq k}\, r s_j^{1/\varrho} > z_{jn}\big\}$$
$$\le \nu\left(\left(\sum\nolimits_{j\neq k} z_{jn}^{\varrho}\right)^{1/\varrho}, \infty\right] \times \upsilon_d\{\boldsymbol{s} \in \mathcal{S}_d\colon s_k = 0\}.$$

The last expression vanishes because $\upsilon_d\{\boldsymbol{s} \in \mathcal{S}_d\colon s_k = 0\} = 0$. In conclusion, μ is ℓ_ϱ-quasinorm symmetric with generator $\Lambda = \mathfrak{W}_d(\nu_{\varrho,1})$.

Conversely, assume that an exponent measure μ on E_d is ℓ_ϱ-quasinorm symmetric with generator Λ_μ. Define a measure μ_ϱ on E_d through the relation $\mu_\varrho(A) = \mu\{\boldsymbol{y} \in E_d\colon \boldsymbol{y}^\varrho \in A\}$ for every Borel set $A \subseteq E_d$, as in the proof of Proposition 3.5. As shown therein, μ_ϱ is an ℓ_1-norm symmetric exponent measure with generator Λ. From Theorem 2 in [9], one can deduce that the image measure of μ_ϱ by $\mathcal{T}_1$ is of the form $\nu^* \times \upsilon_d$, where ν^* is a radial measure with generalized survival function $S_{\nu^*} = \mathfrak{W}_d^{-1}(\Lambda_\mu)$.

Now introduce the radial measure ν with survival function $S_\nu(t) = S_{\nu^*}(t^\varrho)$ for every real $t \in (0,\infty)$, so that $\nu^* = \nu_{\varrho,1}$. For every vector $\boldsymbol{x} \in E_d$, one finds

$$\nu \times \upsilon_d\{(r,\boldsymbol{s}) \in (0,\infty] \times \mathcal{S}_d\colon \mathcal{T}_\varrho^{-1}(r,\boldsymbol{s}) > \boldsymbol{x}\}$$
$$= \nu \times \upsilon_d\{(r,\boldsymbol{s}) \in (0,\infty] \times \mathcal{S}_d\colon \forall_{j\in\{1,\ldots,d\}}\, r s_j^{1/\varrho} > x_j\}$$
$$= \nu \times \upsilon_d\{(r,\boldsymbol{s}) \in (0,\infty] \times \mathcal{S}_d\colon \forall_{j\in\{1,\ldots,d\}}\, s_j > (x_j/r)^\varrho\},$$

where, by the change of variable $r^\varrho \mapsto s$, the last expression equals

$$\int_0^\infty \left(1 - \|\boldsymbol{x}\|_\varrho^\varrho / r^\varrho\right)_+^{d-1} \mathrm{d}\nu(r) = \int_0^\infty \left(1 - \|\boldsymbol{x}\|_\varrho^\varrho / s\right)_+^{d-1} \mathrm{d}\nu^*(s) = \Lambda_\mu(\|\boldsymbol{x}\|_\varrho^\varrho).$$

This shows that the generalized survival function of the image measure of $\nu \times \upsilon_d$ by $\mathcal{T}_\varrho^{-1}$ is precisely S_μ. Because $\mathcal{T}_\varrho$ is one-to-one, one can conclude that the image of μ by $\mathcal{T}_\varrho$ is $\nu \times \upsilon_d$, as claimed.

Proof of Corollary 3.7

Suppose that the generator Λ of the random vector $\boldsymbol{Y} \in \mathcal{M}_\varrho(\upsilon_d)$ is such that $\Lambda(t) \to \infty$ as $t \to 0$. Then the corresponding radial measure ν is unbounded, that is, $\nu(0,\infty] = \infty$. From (2.2), one can see that the distribution of $\boldsymbol{Y}$ is continuous given that the univariate margins of υ_d are Beta distributions, specifically $\mathcal{B}(1, d-1)$. From Lemma 2.6, the unique underlying copula C of $\boldsymbol{Y}$ is the same as that of the max-id random vector $\boldsymbol{Z}$ with ℓ_1-norm symmetric exponent measure with radial measure $\nu_{\varrho,1}$ and the same generator $\Lambda = \mathfrak{W}_d(\nu_{\varrho,1})$. From Corollary 2 in [9], C is reciprocal Archimedean with generator $F = \exp(-\Lambda)$, which concludes the argument.

Proof of Proposition 4.1

From Proposition 2.5, one deduces that, for every vector $\boldsymbol{y} \in [0,\infty)^d$,

$$\Pr(\boldsymbol{Y} \le \boldsymbol{y}) = \exp\left[-\mathrm{E}\left[\gamma\left\{\min_{j\in\{1,\ldots,d\}} \left(y_j / Q_j^{1/\varrho}\right)\right\}^{-\theta}\right]\right] = \exp\left[-\mathrm{E}\left\{\gamma \max_{j\in\{1,\ldots,d\}} \left(Q_j^{\theta/\varrho} / y_j^\theta\right)\right\}\right].$$

Therefore, for every integer $n \in \mathbb{N}$,

$$\{\Pr(\boldsymbol{Y} \leq \boldsymbol{y})\}^n = \exp\left[-\mathrm{E}\left[\gamma \max_{j \in \{1,\ldots,d\}} \{Q_j^{\theta/\varrho}/(y_j/n^{1/\theta})^\theta\}\right]\right] = \Pr(n^{1/\theta}\boldsymbol{Y} \leq \boldsymbol{y}),$$

which shows that the distribution of the random vector $\boldsymbol{Y}$ is max-stable, and hence that it is a multivariate extreme-value distribution. Next, fix an arbitrary integer $j \in \{1, \ldots, d\}$ and observe that from (2.2), one has, for every real $y \in (0, \infty)$,

$$\Pr(Y_j \leq y) = F_j(y) = \exp\{-\mathrm{E}(\gamma Q_j^{\theta/\varrho}/y^\theta)\} = \exp\{-(b_j/y)^\theta\},$$

where $b_j = \{\gamma \mathrm{E}(Q_j^{\theta/\varrho})\}^{1/\theta}$, which shows that the marginal distribution of Y_j is Fréchet with shape parameter θ and scaling parameter b_j, as claimed.

Proof of Proposition 4.2

By Lemma 1.7.6 in [28], the distribution Φ of $d \times \boldsymbol{W}$ is defined, for every Borel subset A of $d\mathcal{S}_d = \{\boldsymbol{x} \in [0,\infty)^d \colon \|\boldsymbol{x}\| = d\}$, by $\Phi(A) = \mathrm{E}\{1_{\mathbb{R}_+ \times A}(\mathbf{Z})(Z_1 + \cdots + Z_d)/d\}$, where $\mathbb{R}_+ \times A = \{y \colon \exists_{s \in A} \exists_{r \in (0,\infty)}\ y = rs\}$. Hence, for every Borel subset B of $\mathcal{S}_d$, one has

$$\varsigma_d(B) = \Phi(dB) = \frac{1}{d} \sum_{j=1}^d \mathrm{E}\{Z_j \times 1_B(\mathbf{Z}/\|\boldsymbol{Z}\|)\},$$

from which the result follows directly, upon recalling that, for every integer $j \in \{1, \ldots, d\}$, $Z_j = Q_j^{\theta/\varrho}/\mathrm{E}(Q_j^{\theta/\varrho})$.

Proof of Proposition 4.3

Fix an integer $j \in \{1, \ldots, d\}$. As discussed prior to the statement of Proposition 4.3, the jth component of the random vector $\boldsymbol{Y}$, denoted Y_j, necessarily has a Fréchet distribution, with location parameter a_j, scale parameter $b_j \in (0, \infty)$, and shape parameter θ_j. Because the support of $\boldsymbol{Y}$ is contained in $[0,\infty)^d$, it must be that $a_j \in [0, \infty)$. However, it turns out that $a_j = 0$. For, if $a_j \in (0,\infty)$, then one would have $F_j(y) = 0$ for all $y \in (0, a_j)$. This would imply that $\mathrm{E}\{S_\nu(y/Q_j{}^{1/\varrho})\} = \infty$, again by (2.2), but this is impossible because

$$\mathrm{E}\{S_\nu(y/Q_j{}^{1/\varrho})\} \leq S_\nu(y) < \infty$$

for any $y \in (0, \infty)$, given that the radial measure ν is Radon.

It thus remains to show that $\theta_1 = \cdots = \theta_d$. Without loss of generality, it suffices to check that the case $\theta_1 > \theta_2$ leads to a contradiction. Because the marginal distributions F_1 and F_2 are Fréchet, one has, for all integers $n \in \mathbb{N}$, $j \in \{1, 2\}$, and reals $y \in \mathbb{R}$, that $F_j^n(yn^{1/\theta_j}) = F_j(y) = \exp\{-(b_j/y)^{\theta_j}\}$. Thus, (2.2) implies that, for all reals $y \in (0, \infty)$,

$$n\mathrm{E}\{S_\nu(yn^{1/\theta_j}/Q_j^{1/\varrho})\} = \mathrm{E}\{S_\nu(y/Q_j^{1/\varrho})\} = (b_j/y)^{\theta_j}. \qquad \text{(A.3)}$$

Now let Q_1^* and Q_2^* be two independent random variables such that Q_1^* has the same distribution as Q_1, and Q_2^* has the same distribution as Q_2. Fix an arbitrary real $y \in (0, \infty)$

and, for each integer $n \in \mathbb{N}$, consider the constant

$$\omega_n = n\mathrm{E}\left\{ S_\nu \left(\frac{yn^{1/\theta_2}}{Q_1^{*1/\varrho} Q_2^{*1/\varrho}} \right) \right\}.$$

Clearly, $\omega_n \in [0, \infty)$ and $\omega_n \le nS_\nu(yn^{1/\theta_2}) < \infty$ because Q_1^* and Q_2^* are bounded above by 1 and the generalized survival function S_ν is non-increasing. Further observe that owing to (A.3), one has

$$\begin{aligned}\omega_n &= \int_0^1 n \int_0^1 S_\nu \left(\frac{yn^{1/\theta_2}}{s^{1/\varrho} t^{1/\varrho}} \right) \mathrm{d}P^{Q_2^*}(t)\, \mathrm{d}P^{Q_1^*}(s) \\ &= \int_0^1 \int_0^1 S_\nu \left(\frac{y}{s^{1/\varrho} t^{1/\varrho}} \right) \mathrm{d}P^{Q_2^*}(t)\, \mathrm{d}P^{Q_1^*}(s) \\ &= \int_0^1 \left(\frac{b_2^\theta s^{\theta/\varrho}}{y^\theta} \right) \mathrm{d}P^{Q_1^*}(s) = \frac{b_2^\theta}{y^\theta}\, \mathrm{E}(Q_1^{\theta/\varrho}),\end{aligned}$$

which is strictly positive, because $Q_1 \not\equiv 0$ almost surely, and does not depend on n. Thus one has $\omega_n = \omega \in (0, \infty)$ for every integer $n \in \mathbb{N}$.

However, at the same time, Fubini's theorem implies that

$$\int_0^1 n \int_0^1 S_\nu \left(\frac{yn^{1/\theta_1 + \epsilon}}{s^{1/\varrho} t^{1/\varrho}} \right) \mathrm{d}P^{Q_1^*}(s)\, \mathrm{d}P^{Q_2^*}(t) = \int_0^1 \int_0^1 S_\nu \left(\frac{yn^{\epsilon}}{s^{1/\varrho} t^{1/\varrho}} \right) \mathrm{d}P^{Q_1^*}(s)\, \mathrm{d}P^{Q_2^*}(t),$$

where $\epsilon \in (0, \infty)$ is such that $1/\theta_2 = 1/\theta_1 + \epsilon$. Calling once again on the fact that the random variables Q_1^* and Q_2^* are bounded above by 1 and that the generalized survival function S_ν is non-increasing, one deduces that the above expression is bounded above by $S_\nu(yn^\epsilon)$, which means that $\omega \le S_\nu(yn^\epsilon)$. However, this is a contradiction because $S_\nu(yn^\epsilon) \to 0$ as $n \to \infty$.

Proof of Proposition 4.4

By assumption, one has that for every integer $j \in \{1, \ldots, d\}$ and real $y \in (0, \infty)$, $F_j(y) = \Pr(Y_j \le y) = \exp\{-(b_j/y)^\theta\}$. Therefore, if $\boldsymbol{b} = (b_1, \ldots, b_d)$ is the vector of marginal scale parameters, then the random vector $(\boldsymbol{Y}/\boldsymbol{b})^\theta$ has a multivariate extreme-value with unit Fréchet margins and spectral measure ς_d. From Proposition 5.11 and the discussion in Section 5.4.1 in [12], one can deduce that $(\boldsymbol{Y}/\boldsymbol{b})^\theta \in \mathcal{M}_1(\varsigma_d)$ and that its radial measure ν^* satisfies $S_{\nu^*}(t) = d/t$ for every real $t \in (0, \infty)$. Therefore, if $\boldsymbol{W}$ is a random vector distributed as ς_d, then for all vectors $\mathbf{y} = (y_1, \ldots, y_d) \in [0, \infty)^d$, one has

$$\begin{aligned}\Pr(\boldsymbol{Y} \le \boldsymbol{y}) &= \Pr\big\{(\boldsymbol{Y}/\boldsymbol{b})^\theta \le (\boldsymbol{y}/\boldsymbol{b})^\theta\big\} \\ &= \exp\left[-\mathrm{E}\left\{ \frac{d}{\min_{j \in \{1,\ldots,d\}} (y_j^\theta / b_j^\theta W_j)} \right\} \right] = \exp\Big[-\mathrm{E}\Big\{ \gamma \max_{j \in \{1,\ldots,d\}} (Q_j^* / y_j^\theta) \Big\} \Big].\end{aligned}$$

Unfortunately, the random vector $\boldsymbol{Q}^* = (Q_1^*, \ldots, Q_d^*)$ is generally not concentrated on $\mathcal{S}_d$. Nevertheless, because ς_d is a spectral measure, one has $\mathrm{E}(W_j) = 1/d$ for every integer

$j \in \{1, \ldots, d\}$, so that

$$\mathrm{E}(\|\boldsymbol{Q}^*\|_1) = \sum_{j=1}^{d} \mathrm{E}(Q_j^*) = \sum_{j=1}^{d} b_j^\theta/\gamma = 1. \tag{A.4}$$

This allows one to define a measure σ_d on $\mathcal{S}_d$ as in statement (i). This is akin to Lemma 1.7.5 in [28]. In view of (A.4), σ_d is a probability measure on $\mathcal{S}_d$.

Now let $\boldsymbol{Q}$ be a random vector distributed as σ_d, fix an arbitrary vector $\boldsymbol{y} = (y_1, \ldots, y_d) \in [0, \infty)^d$ and define a map $f \colon \mathcal{S}_d \mapsto [0, \infty)$ by $f(\boldsymbol{s}) = \gamma \max(s_1/y_1^\theta, \ldots, s_d/y_d^\theta)$ for every vector $\boldsymbol{s} = (s_1, \ldots, s_d) \in \mathcal{S}_d$. Because the map f is non-negative and measurable, it can be approximated by an increasing sequence (f_n) of simple functions on $\mathcal{S}_d$. More precisely, for each integer $n \in \mathbb{N}$, one can find m_n strictly positive reals $\alpha_{1,n}, \ldots, \alpha_{m_n,n} \in (0, \infty)$ and m_n Borel subsets $A_{1n}, \ldots, A_{m_n n}$ of $\mathcal{S}_d$ such that, for every vector $\boldsymbol{s} \in [0, \infty)^d$,

$$f_n(\boldsymbol{s}) = \sum_{i=1}^{m_n} \alpha_{i,n} \mathbf{1}_{A_{i,n}}(\boldsymbol{s})$$

and $f_n \to f$ as $n \to \infty$. Calling on Levi's monotone convergence theorem, one then has, for all reals $y_1, \ldots, y_d \in [0, \infty)$,

$$\begin{aligned} \mathrm{E}\Big\{\gamma \max_{j\in\{1,\ldots,d\}} (Q_j/y_j^\theta)\Big\} &= \int_{\mathcal{S}_d} f(\boldsymbol{s})\, \mathrm{d}\sigma_d(\boldsymbol{s}) = \lim_{n\to\infty} \int_{\mathcal{S}_d} f_n(\boldsymbol{s})\, \mathrm{d}\sigma_d(\boldsymbol{s}) \\ &= \lim_{n\to\infty} \sum_{i=1}^{m_n} \int_{\mathcal{S}_d} \alpha_{i,n}(\mathbf{s}) \Pr\big\{\mathbf{Q} \in A_{i,n}(\mathbf{s})\big\}\, \mathrm{d}\sigma_d(\mathbf{s}), \end{aligned}$$

and the right-hand term can be written alternatively as

$$\begin{aligned} &\lim_{n\to\infty} \sum_{i=1}^{m_n} \alpha_{i,n} \mathrm{E}\left\{\mathbf{1}\left(\frac{\boldsymbol{Q}^*}{\|\boldsymbol{Q}^*\|_1} \in A_{i,n}\right) \|\boldsymbol{Q}^*\|_1\right\} \\ &\qquad = \lim_{n\to\infty} \mathrm{E}\left\{f_n\left(\|\boldsymbol{Q}^*\|_1 \times \frac{\boldsymbol{Q}^*}{\|\boldsymbol{Q}^*\|_1}\right)\right\} \\ &\qquad\qquad = \mathrm{E}\left\{f\left(\|\boldsymbol{Q}^*\|_1 \times \frac{\boldsymbol{Q}^*}{\|\boldsymbol{Q}^*\|_1}\right)\right\}. \end{aligned}$$

Thus, for every vector $\boldsymbol{y} = (y_1, \ldots, y_d) \in [0, \infty)^d$, one has

$$\begin{aligned} \mathrm{E}\Big\{\gamma \max_{j\in\{1,\ldots,d\}} (Q_j/y_j^\theta)\Big\} &= \mathrm{E}\left\{\gamma\, \|\boldsymbol{Q}^*\|_1 \max_{j\in\{1,\ldots,d\}} \left(\frac{Q_j^*}{\|\boldsymbol{Q}^*\|_1 y_j^\theta}\right)\right\} \\ &= \mathrm{E}\Big\{\gamma \max_{j\in\{1,\ldots,d\}} (Q_j^*/y_j^\theta)\Big\} \end{aligned}$$

and hence, upon setting $S_\nu(t) = \gamma\, t^{-\theta}$ for every real $t \in (0, \infty)$, one gets

$$\Pr(\boldsymbol{Y} \le \boldsymbol{y}) = \exp\Big[-\mathrm{E}\Big\{\gamma \max_{j\in\{1,\ldots,d\}} (Q_j/y_j^\theta)\Big\}\Big] = \exp\Big[-\mathrm{E}\big[S_\nu\{\min(\boldsymbol{y}/\boldsymbol{Q}^{1/\theta})\}\big]\Big].$$

This proves that the random vector $\boldsymbol{Y}$ belongs to the class $\mathcal{M}_\theta(\sigma_d)$ and that the radial measure has the desired form.

Proof of Proposition 5.1

Let H be the cumulative distribution function of the random vector $\boldsymbol{Y}$. For every vector $\boldsymbol{x} \in (0,\infty)^d$, one has

$$\lim_{t\to\infty} \frac{1-H(t\boldsymbol{x})}{1-H(t\mathbf{1})} = \lim_{t\to\infty} \frac{\mathrm{E}\Big[S_\nu\{t\min(\boldsymbol{x}/\boldsymbol{Q}^{1/\varrho})\}\Big]}{\mathrm{E}\Big[S_\nu\{t\min(\mathbf{1}/\boldsymbol{Q}^{1/\varrho})\}\Big]} = \lim_{t\to\infty} \frac{\mathrm{E}\Big[S_\nu\{t\min(\boldsymbol{x}/\boldsymbol{Q}^{1/\varrho})\}/S_\nu(t)\Big]}{\mathrm{E}\Big[S_\nu\{t\min(\mathbf{1}/\boldsymbol{Q}^{1/\varrho})\}/S_\nu(t)\Big]}.$$

Given that every component of the random vector $\boldsymbol{Q}$ is bounded above by 1 and that the map S_ν is regularly varying, one can find a constant $\varepsilon \in (0,\infty)$ such that

$$S_\nu\{t\min(\boldsymbol{x}/\boldsymbol{Q}^{1/\varrho})\}/S_\nu(t) \le S_\nu\{t\min(x_1,\ldots,x_d)\}/S_\nu(t) \le \min(x_1,\ldots,x_d)^{-\theta} + \varepsilon$$

for every vector $\boldsymbol{x} \in (0,\infty)^d$ and every sufficiently large real number $t \in \mathbb{R}$. Thus, Lebesgue's dominated convergence theorem implies that, for every vector $\boldsymbol{x} \in (0,\infty)^d$,

$$\lim_{t\to\infty} \mathrm{E}\Big[S_\nu\{t\min(\boldsymbol{x}/\boldsymbol{Q}^{1/\varrho})\}/S_\nu(t)\Big] = \mathrm{E}\big\{\min(\boldsymbol{x}/\boldsymbol{Q}^{1/\varrho})^{-\theta}\big\},$$

and similarly that $\mathrm{E}[S_\nu\{t\min(\mathbf{1}/\boldsymbol{Q}^{1/\varrho})\}/S_\nu(t)] \to 1/\gamma$ as $t \to \infty$.

Thus, one may conclude that, for every vector $\boldsymbol{x} \in (0,\infty)^d$,

$$\lim_{t\to\infty} \frac{1-H(t\boldsymbol{x})}{1-H(t\mathbf{1})} = \gamma\,\mathrm{E}\big[\{\min(\boldsymbol{x}/\boldsymbol{Q}^{1/\varrho})\}^{-\theta}\big] = \mathrm{E}[S_{\nu^*}\{\min(\boldsymbol{x}/\boldsymbol{Q}^{1/\varrho})\}].$$

The expression on the right-hand side is positive for every vector $\boldsymbol{x} \in (0,\infty)^d$, given that

$$\gamma\,\mathrm{E}\big[\{\min(\boldsymbol{x}/\boldsymbol{Q}^{1/\varrho})\}^{-\theta}\big] = \gamma\,\mathrm{E}\big\{\max(\boldsymbol{Q}^{\theta/\varrho}/\boldsymbol{x}^\theta)\big\} \ge \gamma\{\max(\boldsymbol{x})\}^{-\theta}\mathrm{E}\big\{\max(\boldsymbol{Q}^{\theta/\varrho})\big\} > 0,$$

for otherwise $\max(\boldsymbol{Q}^{\theta/\varrho})$ would vanish almost surely, which is a contradiction. Also, for any scalar $c \in (0,\infty)$, one has

$$\mathrm{E}\big[S_{\nu^*}\{\min(c\boldsymbol{x}/\boldsymbol{Q}^{1/\varrho})\}\big] = c^{-\theta}\mathrm{E}\big[S_{\nu^*}\{\min(\boldsymbol{x}/\boldsymbol{Q}^{1/\varrho})\}\big].$$

Hence, the map H is multivariate regularly varying and part (i) of Corollary 5.18 in [12] implies that the random vector $\boldsymbol{Y}$ is in the maximum domain of attraction of the random vector $\boldsymbol{Y}^*$ with distribution function given, for every vector $\boldsymbol{x} \in (0,\infty)^d$, by $\exp\big[-\mathrm{E}[S_{\nu^*}\{\min(\boldsymbol{x}/\boldsymbol{Q}^{1/\varrho})\}]\big]$. The claim then follows from Proposition 2.5.

Proof of Proposition 6.1

It is easily seen that $\zeta = \delta_{(R_1,\boldsymbol{Q}_1)} + \cdots + \delta_{(R_N,\boldsymbol{Q}_N)}$ is a Poisson point process on $(0,\infty] \times \mathcal{S}_d$ with mean measure $\nu \otimes \sigma_d$. Given that the map $\mathcal{T}_\varrho$ is a bijection with $\mathcal{T}_\varrho^{-1}(r,\boldsymbol{q}) = r\boldsymbol{q}^{1/\varrho}$

for every real $r \in (0, \infty)$ and vector $\boldsymbol{q} \in \mathcal{S}_d$, one has

$$\begin{aligned}\Pr\big\{\max\big(\mathbf{0}, R_1\boldsymbol{Q}_1^{1/\varrho}, R_2\boldsymbol{Q}_2^{1/\varrho}, \ldots\big) \leq \boldsymbol{y}\big\} &= \Pr\Big[\zeta\Big\{\mathcal{T}_\varrho\Big([-\boldsymbol{\infty}, \boldsymbol{y}]^{\complement}\Big)\Big\} = 0\Big] \\ &= \exp\big[-\nu \otimes \sigma_d\{\mathcal{T}_\varrho([-\boldsymbol{\infty}, \boldsymbol{y}]^{\complement})\}\big] \\ &= \exp\{-\mu(-\boldsymbol{\infty}, \boldsymbol{y})^{\complement}\},\end{aligned}$$

for all vectors $\boldsymbol{y} \in [0, \infty)^d$. Otherwise, this probability is 0.

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Author contributions

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Conflicts of interest

The authors declare no conflict of interest.

Declaration on the use of artificial intelligence (AI) technologies

The authors declare that no generative AI was used in the preparation of this article.

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