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Parameter estimation of the alpha-stable distribution and applications to financial data

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Abstract

This article introduces a generalized method for estimating parameters of an alpha-stable distribution based on the characteristic function. We propose a novel approach that extends existing techniques. As an application, we compare our proposed method with the maximum likelihood method. Our results demonstrate the efficacy of the proposed method in accurately estimating the parameters of alpha-stable distributions. Additionally, we highlight the advantages of the proposed method ver maximum likelihood method in capturing the tails and skewness of the distribution. This research contributes to the advancement of statistical methods for modeling heavy-tailed and skewed distributed data, with applications in finance, risk management, and other fields.

Keywords: Characteristic function · Financial · Maximum likelihood estimation · Stable distributions · Stock indices

Mathematics Subject Classification: Primary 60E10 · Secondary 60E07.

1. Introduction

Numerous modern finance techniques heavily rely on the assumption that the random variables being investigated follow a Gaussian or normal distribution. However, in finance and other applications, time series often deviate from the Gaussian model because their marginal distributions have heavy tails and are possibly asymmetric. Therefore, it is highly questionable whether the commonly adopted normal assumption is appropriate in such situations.

Financial asset returns are often argued to be the cumulative outcome of a vast number of pieces of information and individual decisions arriving almost continuously in time. Hence, in the presence of heavy tails, it is natural to assume that these returns are approximately governed by a stable non-Gaussian distribution. Other leptokurtic distributions, such as Student-t, Weibull, and hyperbolic, lack the attractive central limit property.

Thus, as the normal distribution fails to describe the empirical evidence in financial markets, an alternative is to introduce stable distributions as proposed by Lévy (1924) and Mandelbrot (1963). Stable distributions have been successfully fitted to stock returns, excess bond returns, foreign exchange rates, commodity price returns, and real estate returns McCulloch (1996), Mittnik et al. (2000). However, several studies have provided strong evidence for the stable model versus the Gaussian model (Weron, 1995; Nolan, 2020, 2021).

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The stable distributions can account for heavy tails and asymmetrical behavior, and depending on four parameters, these distributions are more flexible than the normal distribution for fitting empirical data. Another good property is that these distributions have an attraction domain, meaning they are limits of sums of random variables. Several methods for simulating stable distributed random variables and estimating their parameters are available. Lévy-Véhel and Walter (2002) highlighted the advantages of stable distributions for financial modeling and provided methods for simulating stable distributed random variables and estimating their parameters. Studies of estimation methods for α-stable distributions are available, but to our knowledge, these studies focus on certain classes of estimators. Mittnik et al. (1999) compared the Fourier transform and direct calculation methods of the probability density function (PDF) of stable distributions and Kogon and Williams (1998) focussed on characteristic function methods (Weron, 1995).

Estimating parameters of the stable distribution is severely affected by the lack of an explicit form for the PDF, then resorting to numerical methods that can be complex and slow to execute. A large number of algorithms, more or less efficient and more or less fast, are proposed based on different approaches. We can mention the empirical quantile approach McCulloch (1986)), the empirical characteristic function approach (Koutrouvelis, 1981; Kogon and Williams, 1998), and the maximum likelihood (ML) approach (DuMouchel, 1973; Nolan, 2001). In general, the performance of all these methods is good, but Ojeda (2001) observed, in a comparative study, that ML methods provide the most accurate estimates albeit with longer execution times. This is confirmed by simulation studies conducted by Weron and Weron (1995). There are four main families of estimators that give rise to several methods based on ML, quantiles, characteristic function, and least squares.

Estimating the α-stable parameters using the characteristic function allows us to model extreme phenomena in scientific domains, including finance, meteorology, physics, and engineering (Uchaikin and Zolotarev, 1999; Nolan, 2003, 2014, 2020). Characteristic function-based methods are fast, efficient, and robust, and continue to be developed to improve their accuracy and applicability in practical contexts.

The first one who explored the idea of estimating the parameters of the α-stable law based on the characteristic function was Press (1972), but then, several modifications have been proposed by Koutrouvelis (1980), Koutrouvelis (1981), Kogon and Williams (1998) and Krutto (2016). Consequently, Koutrouvelis (1980), Kogon and Williams (1998) and Krutto (2016) used much more than $k = 2$ points in their estimation implementation algorithm (although their estimator are expressed based on 2 points). And all these versions deal in this case or even are very restricted and we have extended these versions with $\forall k \geq 2$ that we will give more details in the continuation of work. Another advantage of these methods is that they can be extended to cases which are not independent and identically distributed, in particular dynamic heteroscedastic models: we can consider multivariate or conditional multivariate or conditional characteristic functions. The asymptotic properties (convergence and normality) are preserved (Feuerverger and McDonnouch, 1981; Knight and Yu, 2002).

The objective of this article is to propose a generalized version of parameter estimation methods for an α-stable distribution based on the characteristic function. In Section 2, we present the main characteristics of α-stable distributions and their use in modeling complex phenomena such as price fluctuations in financial assets. Section 3 describes the methodology we use to estimate the α-stable parameters. In Sections 4, a simulation study is provided to evaluate the statistical performance of the proposed estimation method. In Sections 5 and 6, we apply our results to stock markets to model price and interest rate fluctuations. We compare the results obtained with other models used in the literature. In Section 7, we conclude by emphasizing the use of α-stable distributions in financial modeling and proposing future research, including the application of these distributions to other types of financial data and the exploration of new estimation methods to improve model accuracy.

2. Preliminaries and notations

In this section, we delve into the fundamental basics of α-stable distributions, examining their definitions, fundamental properties, as well as some existing results. α-stable distributions represent a class of probability distributions characterized by their stability under convolution and their ability to model complex and asymmetric phenomena Paulo H. et al. (2019). Their relevance extends to a variety of fields, from finance to geophysics, and telecommunication. Let X be a random variable with α-stable distribution. Then, if $\forall(a,b) \in \mathrm{R}_+^* \times \mathrm{R}_+^*$, $\exists c > 0$ and $k \in \mathrm{R}$, such that $aX_1 + bX_2 \overset{\mathrm{d}}{=} cX + k$, where X_1 and X_2 are independent copies of X and $\overset{\mathrm{d}}{=}$ denotes convergence in distribution. Note that, if $k = 0$, the distribution is strictly stable. Equivalently, a random variable X is said to have a α-stable distribution if and only if, for any integer $n \geq 1$ and for any family $X_1, \ldots, X_n$ of independent and identically distributed random variables of the same law as X, there exist two real numbers $a_n > 0$ and b_n such that $[(X_1 + \cdots + X_n) - b_n]/a_n \overset{\mathrm{d}}{=} X$. Variables with a Levy-stable distribution have the disadvantage of not having (except in three cases) explicit forms for the PDF and CDF. A random variable X with a stable distribution is typically described by its characteristic function Δ_X (inverse Fourier transform of the PDF f) defined on R by $\Delta_X(t) = \mathrm{E}\big(\exp(\mathrm{i}tX)\big) = \int_{-\infty}^{+\infty} \exp(\mathrm{i}tx) f(x)\mathrm{d}x$, and having representations according to the different parameterizations of the α-stable distributions, where $\mathrm{i} = \sqrt{-1}$ is the imaginary number. The most famous of these representations is given in Samorodnitsky (1994) by

$$\Delta_X(t) = \mathrm{E}\left[\exp(\mathrm{i}tx)\right] = \exp\left(\mathrm{i}\mu t - g_{\alpha,\beta,\sigma}(t)\right), \tag{2.1}$$

where

$$g_{\alpha,\beta,\sigma}(t) = \begin{cases} \sigma^\alpha |t|^\alpha \left[1 - i\beta \mathrm{sign}(t) \tan\left(\frac{\pi\alpha}{2}\right)\right], & \text{if } \alpha \neq 1 \\ \sigma^\alpha |t|^\alpha \left[1 + \frac{2}{\pi} i\beta \mathrm{sign}(t)(\log|t|)\right], & \text{if } \alpha = 1 \end{cases}, \quad \mathrm{sign}(t) = \begin{cases} 1, & \text{if } t > 0 \\ 0, & \text{if } t = 0 \\ -1, & \text{if } t < 0. \end{cases}$$

The α-stable distribution is thus characterized by four real parameters α, β, μ and σ. The parameter α, called characteristic exponent or stability index, is an indicator of the degree of thickness of the tails of the distribution: the smaller it is, the thicker the tails are which corresponds to very large fluctuations. It is the most important parameter, it is between 0 and 2 ($0 < \alpha \leq 2$). Its maximum value $\alpha = 2$, corresponds to a particular α-stable distribution: the Gaussian distribution or normal distribution. β is the parameter of dissymmetry, it varies between -1 and 1 ($-1 \leq \beta \leq 1$) and when it is null, the distribution is symmetrical with respect to μ.

When α approaches to two, β loses its effect leading to a trend towards the normal distribution. The parameters $\mu \in \mathrm{R}$ and $\sigma > 0$ represent the usual characteristics of position and scale respectively with the remark that for the Gaussian distribution, the standard deviation is $\sigma\sqrt{2}$. A random variable X of α-stable distribution will be noted, according to Samorodnitsky (1994), by $X \sim S_\alpha(\beta, \mu, \sigma)$. The three exceptions mentioned above are the very famous Gaussian distribution $S_2(0, \mu, \sigma)$, the less known Cauchy distribution $S_1(0, \mu, \sigma)$ and the Lévy distribution $S_{1/2}(1, \mu, \sigma)$. The α-stable distribution has an additivity property according to which the sum of two independent stable random variables of the same stability index α is still stable with the same characteristic exponent α. This very interesting property is used in finance to study portfolios where two assets with the same value for α can be considered together. Figure 1 illustrates the influence of each parameter of the α-stable distribution on its PDF.

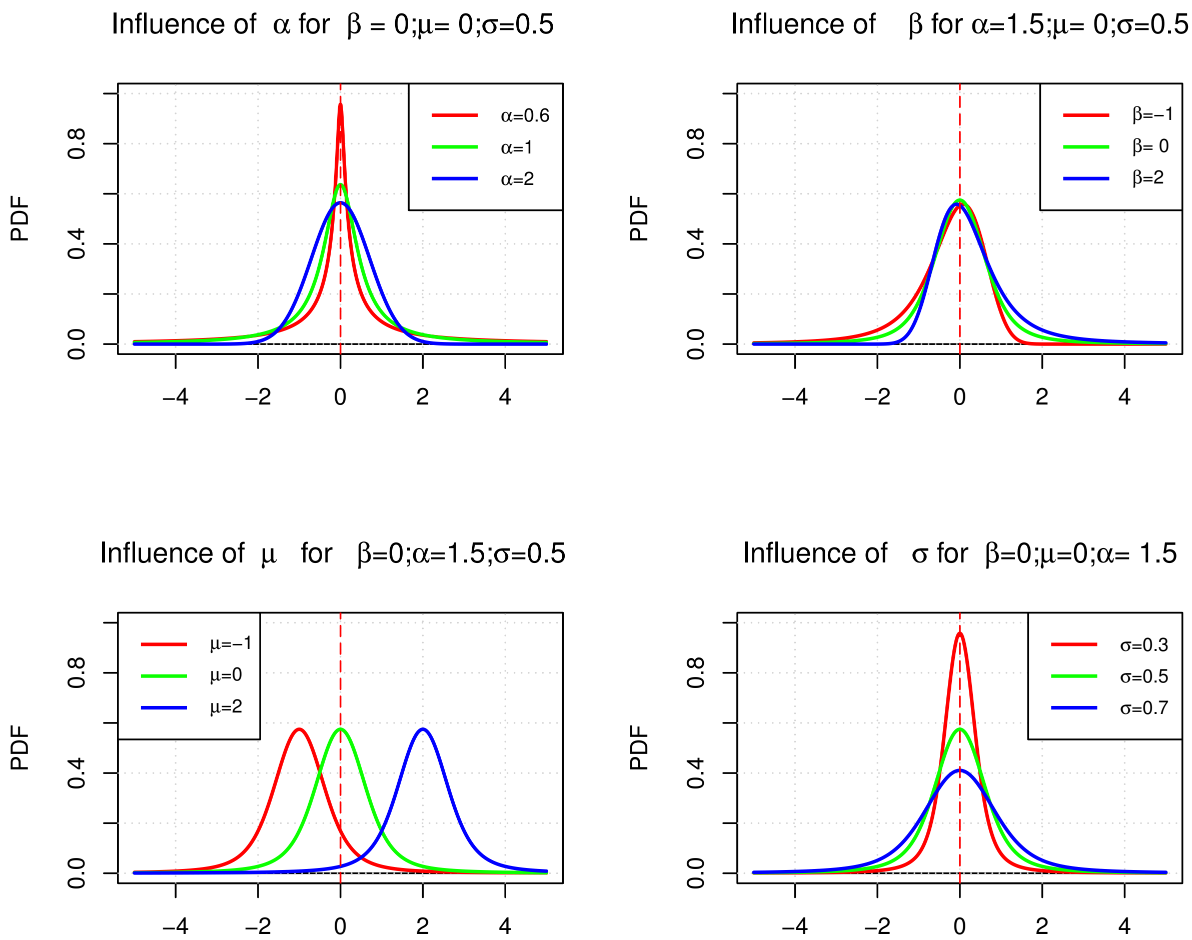

Figure 1.: Plots of influence of the parameters of the α-stable distribution on its PDF.

One of the particularities of the stable distribution is that it has infinite variance if α is strictly less than two. In fact, the moments of order p of $X \sim S_\alpha(\beta, \mu, \sigma)$ are such that for $\alpha = 2$, $\mathrm{E}(|X|^p) < +\infty$, $\forall p \in \mathrm{N}$, defined as

$$\mathrm{E}(|X|^p) = \begin{cases} < \infty; & \text{if } 0 < p < \alpha; \\ = \infty; & \text{if } p \geq \alpha. \end{cases}$$

When the characteristic exponent of a random variable X is strictly less than two, the variance is infinite and the tails variance is infinite, with the tails being asymptotically equivalent to that of a Pareto distribution. More precisely, it is shown that (Samorodnitsky, 1994)

$$\lim_{t\to\infty} \; t^\alpha P(X > t) = C_\alpha \tfrac{1+\beta}{2} \sigma^\alpha; \tag{2.2}$$

$$\lim_{t\to\infty} t^\alpha P(X < -t) = C_\alpha \tfrac{1-\beta}{2} \sigma^\alpha; \tag{2.3}$$

where

$$C_\alpha = \left(\int_0^\infty x^{-\alpha}\sin(x)\mathrm{d}x\right)^{-1} \begin{cases} \frac{1-\alpha}{\Gamma(2-\alpha)\cos\left(\frac{\pi\alpha}{2}\right)}; & \text{if } \alpha \neq 1, \\ 2/\pi; & \text{if } \alpha = 1; \end{cases}$$

with $\Gamma(\theta)$ being the Euler gamma function defined for $\theta > 0$, by $\Gamma(\theta) = \int_0^{+\infty} x^{\theta-1}\exp(-x)\mathrm{d}x$. Equations (2.2) and (2.3) lead to the fact that, when $n \to \infty$, and $\mathrm{P}(X > t) \cong C_\alpha \sigma^\alpha t^{-\alpha}$.

Hence, we see that the α-stable distribution takes into consideration the distribution tails which often carry essential information, whereas the Gaussian distribution neglects these tails, thus leading to an error which can be fatal for the investor. The disadvantage of the characteristic function defined in Equation (2.1) is that it is not continuous if $\alpha = 1$, which makes it not adapted to numerical calculations. For these reasons, Zolotarev (1998) proposed another parameterization called S_α^0, which is usable for numerical calculations. To simulate α-stable distributions, there is an algorithm developed by Chambers et al. (1976), which allows us to generate random numbers from the $S_\alpha(\beta, 0, 1)$ distribution. To obtain an $S_\alpha(\beta, \mu, \sigma)$ distribution, with $\alpha \in]0, 2]$ and $\beta \in [-1, 1]$, the parameters α and σ must be estimated. These parameters can be correctly estimated by the method of McCulloch (1986) for small values of β, which is often the case for stock exchange chronicles.

3. Methodology and main results

In this section, we present the theoretical results obtained in this article, which focuses on methods for estimating the parameters of an α-stable distribution. These theoretical results stem from a thorough analysis and meticulous exploration of the fundamental properties of the α-stable distribution, as well as the techniques and methods used to estimate its parameters. Let $\tilde{\Phi}_X$ the empirical characteristic function of Φ_X which is defined by

$$\tilde{\Phi}_X(t) = \frac{1}{n}\sum_{j=1}^{n} \exp(\mathrm{i}tx_j).$$

It is assumed that $X_1, \dots, X_n$ are independent and identically distributed random variables with the same α-stable distribution. The characteristic function $\tilde{\Phi}_X$ is an unbiased estimator of Φ_X because

$$\mathrm{E}\Big[\tilde{\Phi}_X(t)\Big] = \mathrm{E}\Bigg[\frac{1}{n}\sum_{j=1}^{n}\exp(\mathrm{i}tx_j)\Bigg] = \frac{1}{n}\sum_{j=1}^{n}\mathrm{E}\left[\exp(iix_j)\right] = \mathrm{E}\left[\exp(\mathrm{i}tx)\right] = \Phi_X(t).$$

Since $|\tilde{\Phi}_X(t)| \leq 1$, then all moments of $\tilde{\Phi}_X(t)$ are finite. From the law of large numbers, $\tilde{\Phi}_X(t)$ is an unbiased estimator of $\Phi_X(t)$, the theoretical characteristic function. The estimation methods in this work are methods based on this expression.

Each of the methods tries to obtain the characteristic function of a stable random variable closer to the empirical characteristic function in some sense. These methods are justified by the one-to-one correspondence that exists between the distribution functions and their Fourier-Stieltjes transforms. According to Euler formula for complex numbers we have

$$\exp(\mathrm{i}x) = \cos(x) + \mathrm{i}\sin(x). \tag{3.4}$$

Thus, using the formula given in Equation (3.4), the empirical characteristic function is written as

$$\tilde{\Phi}_X(t) = \sum_{j=1}^{n} \frac{\exp(\mathrm{i}tx_j)}{n} = \sum_{j=1}^{n} \frac{(\cos(tx_j) + \sin(tx_j))}{n} = \underbrace{\frac{1}{n}\sum_{j=1}^{n}\cos(tx_j)}_{a} + \mathrm{i}\underbrace{\frac{1}{n}\sum_{j=1}^{n}\sin(tx_j)}_{b} = a + \mathrm{i}b.$$

Denote θ and r as the argument and modulus of $\tilde{\Phi}_X(t)$, respectively. Then, we have that

$$\tilde{\Phi}_X(t) = r\exp(\mathrm{i}\theta) \rightarrow \log\left(\tilde{\Phi}_X(t)\right) = \log(r) + \mathrm{i}\theta.$$

Denote by $\Re\big(\log(\tilde{\Phi}_X(t))\big)$ and $\Im\big(\log(\tilde{\Phi}_X(t))\big)$ the real and imaginary parts of $\log\left(\tilde{\Phi}_X(t)\right)$, respectively. Therefore, we get

$$\begin{aligned} \Re\big(\log(\tilde{\Phi}_X(t))\big) &= \log(r) = \log\left(\sqrt{a^2+b^2}\right) \\ &= \frac{1}{2}\log\left[\left(\frac{1}{n}\sum_{j=1}^{n}\cos(tx_j)\right)^2 + \left(\frac{1}{n}\sum_{j=1}^{n}\sin(tx_j)\right)^2\right], \end{aligned} \tag{3.5}$$

$$\Im(\log(\tilde{\Phi}_X(t))) = \theta = \mathrm{atan2}(b,a) = \mathrm{atan2}\left(\frac{1}{n}\sum_{j=1}^{n}\sin(tx_j), \frac{1}{n}\sum_{j=1}^{n}\cos(tx_j)\right), \tag{3.6}$$

where the function atan2 is defined by

$$\mathrm{atan2}(y,x) = \begin{cases} \arctan\left(\frac{y}{x}\right), & \text{if } x > 0; \\ \arctan\left(\frac{y}{x}\right) + \pi, & \text{if } x < 0, y \geq 0; \\ \arctan\left(\frac{y}{x}\right) - \pi, & \text{if } x < 0, y < 0; \\ \frac{+\pi}{2}, & \text{if } x = 0, y > 0; \\ \frac{-\pi}{2}, & \text{if } x = 0, y < 0; \\ \text{undefined}, & \text{if } x = 0, y = 0. \end{cases}$$

Theorem 1. Let X be an α-stable distributed random variable, denoted by $X \sim S_\alpha(\beta,\sigma,\mu)$, and $t_1,\ldots,t_n \in \mathrm{R}_+^*$ with $n \geq 2$. Then, for $\alpha \neq 1$, the parameters of the distribution of X can be estimated by

$$\begin{aligned} \log(\hat{\sigma}) &= \frac{B\log\left(-\Re\big(\log(\tilde{\Phi}_X(t_k))\big)\right) - A\log(t_k)}{A - n\log\left(-\Re\big(\log(\tilde{\Phi}_X(t_k))\big)\right)}, \quad \hat{\alpha} = \frac{A}{B + n\log(\hat{\sigma})}, \\ \hat{\mu} &= \frac{C_3\Im\big(\log(\tilde{\Phi}_X(t_k))\big) - C_1 t_k^{\hat{\alpha}}}{C_3 t_k - C_2 t_k^{\hat{\alpha}}}, \quad \hat{\beta} = \frac{C_1 - \hat{\mu}C_2}{C_3\hat{\sigma}^{\hat{\alpha}}\tan\left(\frac{\pi\hat{\alpha}}{2}\right)}, \end{aligned} \tag{3.7}$$

where

$$\begin{aligned} A &= \sum_{j=1}^{n}\log\left(-\Re(\log\tilde{\Phi}_X(t_j))\right); \quad B = \sum_{j=1}^{n}\log(t_j); \\ C_1 &= \sum_{j=1}^{n}\Im\big(\log\tilde{\Phi}_X(t_j)\big); \quad C_2 = \sum_{j=1}^{n} t_j; \quad C_3 = \sum_{j=1}^{n} t_j^{\hat{\alpha}}; \quad t_k \in \{t_1,\ldots,t_n\}, \end{aligned} \tag{3.8}$$

where $\Im\big(\log(\tilde{\Phi}_X(t))\big)$ and $\Re\big(\log(\tilde{\Phi}_X(t))\big)$ are defined in Equations (3.6) and (3.5) respectively. See proof of this theorem in Appendix A.

THEOREM 2. Let X be an α-stable distributed random variable and $t_1, \dots, t_n \in \mathrm{R}_+^*$, for all $n \geq 2$. For $\alpha = 1$, the three other parameters of the distribution of X can be estimated by the relations stated as

$$\hat{\sigma} = \frac{-\sum_{j=1}^{n} \Re\big(\log \tilde{\Phi}_X(t_j)\big)}{C_2}; \quad \hat{\beta} = \frac{\pi}{2} \frac{C_2 \Im\big(\log \tilde{\Phi}_X(t_k)\big) - A_1 t_k}{\hat{\sigma} t_k \left[A_2 - C_2 \log t_k\right]}; \quad \hat{\mu} = \frac{A_1 + \frac{2}{\pi}\hat{\sigma}\hat{\beta} A_2}{C_2}, \quad (3.9)$$

with

$$A_1 = \sum_{j=1}^{n} \Im(\log \tilde{\Phi}_X(t_j)), \quad A_2 = \sum_{j=1}^{n} t_j \log t_j, \quad C_2 = \sum_{j=1}^{n} t_j, \quad t_k \in \{t_1, \dots, t_n\}, \qquad (3.10)$$

where $\Im\big(\log(\tilde{\Phi}_X(t))\big)$ and $\Re\big(\log(\tilde{\Phi}_X(t))\big)$ are defined in Equations (3.6) and (3.5) respectively. See proof of this theorem in Appendix B.

The following steps allow us to generate α-stable distributed random numbers based on Theorems 1 and 2:

1: Collect the data $y_1, \dots, y_m$.
2: Choose $t_1, \dots, t_n$ as strictly positive real numbers.
3: Calculate the real and imaginary parts of $\log\big(\tilde{\Delta}_Y\big)$ from Equations (3.5) and (3.6):

$$\Re\big(\log(\tilde{\Delta}_Y(t_i))\big) \longleftarrow \frac{1}{2} \log\left[\left(\frac{1}{m}\sum_{j=1}^{m} \cos(t_i y_j)\right)^2 + \left(\frac{1}{n}\sum_{j=1}^{m} \sin(t_i y_j)\right)^2\right].$$

$$\Im\big(\log(\tilde{\Delta}_Y(t_i))\big) \longleftarrow \mathrm{atan2}\left(\frac{1}{m}\sum_{j=1}^{m} \sin(t_i y_j), \frac{1}{m}\sum_{j=1}^{m} \cos(t_i y_j)\right).$$

4: Obtain coefficients A, B, C_1, C_2, C_3, A_1, and A_2 from Equations (3.8) and (3.10) as

$$A \longleftarrow \sum_{j=1}^{n} \log\big(-\Re\big(\log \tilde{\Phi}_X(t_j)\big)\big); \quad B \longleftarrow \sum_{j=1}^{n} \log(t_j);$$

$$C_1 \longleftarrow \sum_{j=1}^{n} \Im\big(\log \tilde{\Phi}_X(t_j)\big); \quad C_2 \longleftarrow \sum_{j=1}^{n} t_j; \quad C_3 \longleftarrow \sum_{j=1}^{n} t_j^{\alpha};$$

$$A_1 \longleftarrow \sum_{j=1}^{n} \Im\big(\log \tilde{\Phi}_X(t_j)\big); \quad A_2 \longleftarrow \sum_{j=1}^{n} t_j \log t_j.$$

We take $t_k = \min\{t_1, \dots, t_n\}$ or $t_k = (t_1 + \cdots + t_n)/n$.

5: If $\hat{\alpha}$ takes a value equal to one, then use Theorem 2 (from Equation (3.9)) and consider

$$\hat{\sigma} \longleftarrow \frac{-\sum_{j=1}^{n} \Re\big(\log \tilde{\Phi}_X(t_j)\big)}{C_2}; \quad \hat{\beta} \longleftarrow \frac{\pi}{2} \frac{C_2 \Im\big(\log \tilde{\Phi}_X(t_k)\big) - A_1 t_k}{\hat{\sigma} t_k \left[A_2 - C_2 \log t_k\right]}; \quad \hat{\mu} \longleftarrow \frac{A_1 + \frac{2}{\pi}\hat{\sigma}\hat{\beta} A_2}{C_2}.$$

6: Else go to the Theorem 1 (from Equation (3.7)) and utilize

$$\log(\hat{\sigma}) \longleftarrow \frac{B\log\big(-\Re(\log(\tilde{\Phi}_X(t_k)))\big) - A\log(t_k)}{A - n\log(-\Re\big(\log(\tilde{\Phi}_X(t_k))\big))}; \quad \hat{\alpha} \longleftarrow \frac{A}{B + n\log(\hat{\sigma})};$$

$$\hat{\mu} \longleftarrow \frac{C_3\Im\big(\log(\tilde{\Phi}_X(t_k))\big) - C_1 t_k^{\hat{\alpha}}}{C_3 t_k - C_2 t_k^{\hat{\alpha}}}; \quad \hat{\beta} \longleftarrow \frac{C_1 - \hat{\mu}C_2}{C_3\hat{\sigma}^{\hat{\alpha}}\tan(\frac{\pi\hat{\alpha}}{2})}.$$

4. Simulation study

When evaluating the effectiveness of a parameter estimator, particularly in cases where simulation from a given distribution is straightforward, one common approach is to analyze its performance across a range of simulated datasets. This entails selecting parameter values that cover a wide spectrum and varying sample sizes to explore both finite sample properties and the method asymptotic behavior. By generating diverse sets of simulated data, one can observe how well the estimator performs under different conditions. Statistical indicators can then be computed and summarized in tables for comparison. This comparative analysis allows for a thorough examination of various estimation techniques or a focused assessment of a single method across the entire parameter space. For an illustrative example of such simulation studies concerning the estimation of parameters for stable distributions, interested readers can refer to the work by Weron (1995).

In this section, we conduct numerical simulations to assess the performance of our novel approach and compare it with other analytical methods proposed in the literature. Specifically, we examine the classical ML method (see Nolan, 2001) for estimating the four parameters of the α-stable PDF, as well as for estimating these parameters from samples.

To quantitatively evaluate the performance of the parameter estimators, we employ several criteria, including the mean squared error (MSE), standard error (SE), and standard deviation. Throughout our simulations, we denote by M the number of times an experiment is repeated, and we fix M at 200 for all simulations. This choice ensures robustness and reliability in our assessment of the various estimation methods. The data is produced using the approach introduced by Chambers et al. (1976). The ML method described earlier is implemented in the `STABLE` package, developed by Nolan for `R` RobustAnalysisInc. (2010) and RobustAnalysisInc. (2013). All associated simulations are conducted utilizing this package.

To initially assess the performance of our proposed estimator, we employ the MSE criterion, standard deviation (S_n), SE, which are defined as

$$\text{MSE}(\hat{\theta}) = \frac{1}{M}\sum_{i=1}^{M}|\theta - \hat{\theta}_i|^2; \quad S_n(\hat{\theta}) = \left(\frac{n}{M}\sum_{i=1}^{M}\left(\hat{\theta}_i - \bar{\theta}\right)^2\right)^{\frac{1}{2}}; \quad \text{SE}(\hat{\theta}) = \frac{S_n(\hat{\theta})}{\sqrt{n}},$$

where $\hat{\theta}_i, i = 1, \ldots, M$ is the estimated parameter in every simulation runs, n is the sample size, M is the number of simulations and $\bar{\theta}$ is the sample mean of the estimators $\hat{\theta}_i$, for $i = 1, \ldots, M$. In each simulation run, we obtain estimates $\hat{\alpha}_i$, $\hat{\beta}_i$, $\hat{\mu}_i$ and $\hat{\sigma}_i$ for the parameters of interest. The MSE is a used criterion for evaluating the performance of estimators, providing a straightforward measure of their accuracy. We conduct numerical simulations using synthetic observations generated from an α-stable distribution, covering a broad spectrum of parameter combinations. Specifically, we consider two scenarios: $n = 10^3$ and $n = 10^4$ independent and identically distributed (IID) samples from the α-stable distribution, where α takes values of $\alpha = 0.2, 0.5, 0.75, 1, 1.25, 1.5, 1.75$ and $\beta = -0.5, 0.25, 0.4, -0.75, 0.5, 0.6, 1$; M $= 200$ replications; $t_1 = 0.03, t_2 = 0.04$ and $t_3 = 0.09$.

Table 1.: Performance of the proposed estimator.

α	β	n	mean($\hat{\alpha}$)	min($\hat{\alpha}$)	max($\hat{\alpha}$)	$S_n(\hat{\alpha})$	MSE($\hat{\alpha}$)	SE($\hat{\alpha}$)	failure	time
0.20	-0.50	10^3	0.203	0.133	0.269	0.755	0.001	0.002	0	2.28
		10^4	0.201	0.175	0.217	0.768	0.000	0.001	0	14.73
0.50	0.25	10^3	0.491	0.422	0.584	0.972	0.001	0.002	0	1.69
		10^4	0.500	0.467	0.525	1.063	0.000	0.001	0	13.32
0.75	0.40	10^3	0.744	0.639	0.829	1.026	0.001	0.002	0	3.23
		10^4	0.751	0.718	0.778	1.121	0.000	0.001	0	26.81
1.00	-0.75	10^3	0.062	0.000	0.490	2.782	0.888	0.006	1	2.77
		10^4	0.986	0.941	1.027	1.429	0.000	0.001	0	29.48
1.25	0.50	10^3	1.247	1.111	1.362	1.406	0.002	0.003	0	0.06
		10^4	1.250	1.205	1.288	1.486	0.000	0.001	0	0.12
1.50	0.60	10^3	1.495	1.365	1.616	1.528	0.002	0.003	0	0.45
		10^4	1.500	1.448	1.534	1.573	0.000	0.001	0	3.36
1.75	1.00	10^3	1.747	1.594	1.888	1.510	0.002	0.003	0	0.62
		10^4	1.750	1.708	1.799	1.604	0.000	0.001	0	6.57

Table 2.: Performance of ML estimation.

α	β	n	mean($\hat{\alpha}$)	min($\hat{\alpha}$)	max($\hat{\alpha}$)	$S_n(\hat{\alpha})$	MSE($\hat{\alpha}$)	SE($\hat{\alpha}$)	failure	time
0.20	-0.50	10^3	0.203	0.144	0.266	0.691	0.000	0.002	0	0.83
		10^4	0.201	0.180	0.220	0.656	0.000	0.000	0	4.21
0.50	0.25	10^3	0.501	0.438	0.569	0.823	0.001	0.002	0	0.53
		10^4	0.500	0.477	0.525	0.910	0.000	0.001	0	2.92
0.75	0.40	10^3	0.748	0.663	0.823	0.949	0.001	0.002	0	0.52
		10^4	0.750	0.720	0.772	1.013	0.000	0.001	0	2.75
1.00	-0.75	10^3	0.881	0.766	1.020	1.491	0.016	0.003	0	0.55
		10^4	0.876	0.836	0.922	1.424	0.015	0.001	0	2.64
1.25	0.50	10^3	1.243	1.091	1.343	1.345	0.002	0.003	0	0.62
		10^4	1.250	1.205	1.281	1.415	0.000	0.001	0	2.75
1.50	0.60	10^3	1.497	1.367	1.659	1.607	0.003	0.004	0	0.52
		10^4	1.500	1.459	1.531	1.461	0.000	0.001	0	3.06
1.75	1.00	10^3	1.734	1.632	1.884	1.401	0.002	0.003	0	0.61
		10^4	1.746	1.705	1.791	1.416	0.000	0.001	0	3.10

We explore all possible combinations of α and β. Additionally, we set $\sigma = 1$ and $\mu = 0.25$ for the dispersion and location parameters, respectively. The resulting means and MSEs obtained by the estimators across 200 simulations are summarized in Table 1 and 2. This comprehensive analysis allows us to assess the performance of our proposed estimator relative to the classical ML method and provides insights into its effectiveness under various conditions.

To compare the two methods presented in Table 1 and 2, we can examine several aspects such as the estimated mean, MSE, standard deviation, execution time (time), and failure count (failure). Firstly, we observe that the estimated means (mean) for both methods are generally close to each other, indicating similarity in the performance of the estimators. Regarding the MSE, we find that, in most cases, the proposed method has lower MSE values than the ML method. This suggests that the proposed method tends to provide more precise parameter estimates. Concerning the standard deviation (S_n), the results are similar to those of the MSE. The proposed method appears to provide more precise estimates with lower standard deviations. As for the execution time (time), we observe that the proposed method seems to require more computation time than the ML method in most cases. This may be due to the increased complexity of the algorithm used in the proposed method. Regarding the failure count (failure), we observe that it is generally zero for both methods, indicating overall satisfactory performance in most cases. The proposed method appears to offer similar or better performance than the ML method in terms of estimation accuracy, but it may require more computation time.

5. Application to stock indices

The objective of this section is to adjust the evolution of some of the world most well-known stock indices using proposed methods and then to compare the adjustment with the ML method. Before starting this adjustment, we introduce the following definition: A stock index is a statistical measure used to represent the performance of the stock prices of a given financial market. Stock indices are generally calculated based on the weighted value of the stock prices of the main companies listed on a stock exchange in a given country or region. The evolution of the stock index is often used as an indicator of the overall state of the stock market. Investors can use stock indices to evaluate market performance and make investment decisions based on the evolution of these indices. Well-known examples of stock indices include the Dow Jones Industrial Average (DJIA) in the United States, the FTSE 100 in the UK, and the Nikkei 225 in Japan. The data used in this section is available on the Yahoo Finance (https://finance.yahoo.com, accessed on 13 April 2024), and we conducted our simulation based on the period from 22 February 2022 and 22 February 2023.

Before exploring how to apply our estimator method with their different approaches to this portfolio, we conduct a technical analysis of the four stocks in our portfolio. Technical analysis is widely used by financial analysts to comprehend financial markets and predict their future movements. It is based on the assumption that the past performance of a financial asset provides the best information about its future performance. One could say that technical analysis relies on three principles: the market itself provides enough information to deduce its trends; prices move according to determined trends, movements, or rules; and past events will repeat in the future.

Analyzing the price movements of a financial asset is crucial for understanding market fluctuations. It offers a general overview of transaction and price developments, enabling the anticipation of future trends. Therefore, we conduct a detailed graphical analysis of the of stock indices include the DJIA in the United States, the FTSE 100 in the UK, and the Nikkei 225 in Japan.

Figure 2 and Table 3 provide an overview of descriptive statistics for the financial data of each stock during the study period. In the histograms at the top of Figure 2 , we can observe that the data from these portfolios are not stationary. According to this Figure 2, we can say that, despite apparent fluctuations both upward and downward, the value of each of the portfolios experienced three phases. Taking the case of CAC 40, we have the following observations: a phase of strong growth; a phase of significant decline, it is noticeable that the value exhibited a downward trend during this period; and phase of instability marked by fluctuations.

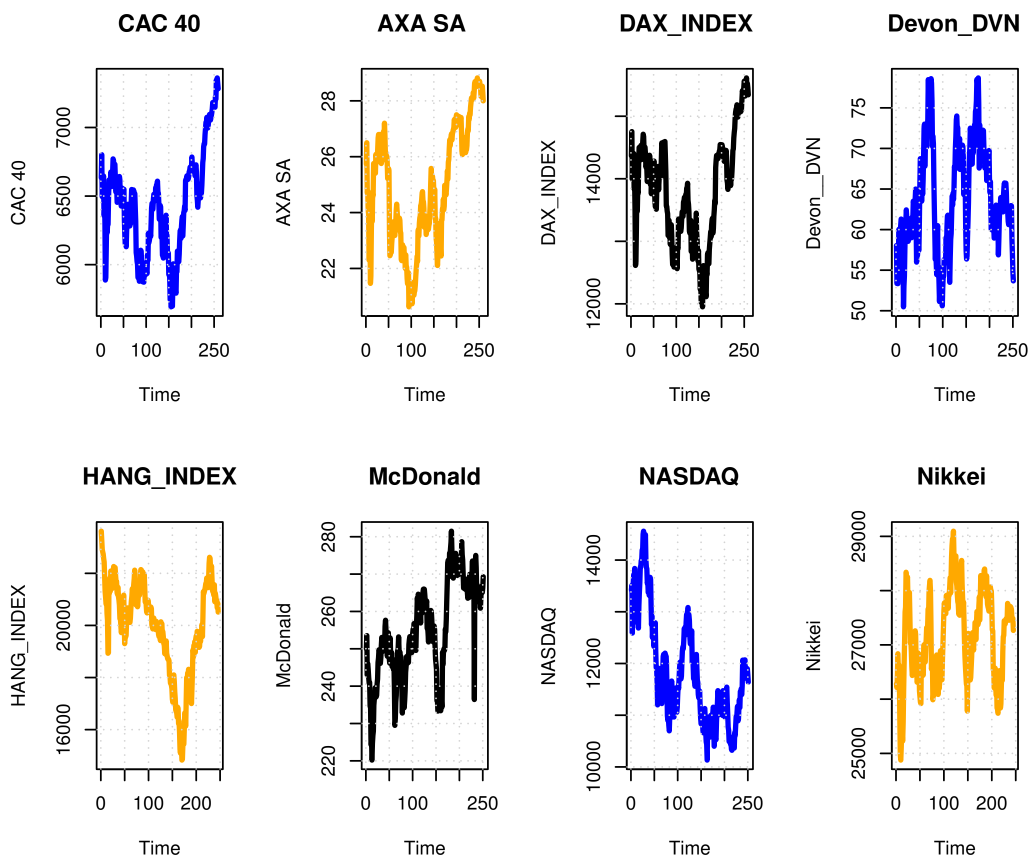

Figure 2.: Evolution of Stock indices the period from 22 February 2022 to 22 February 2023.

If we have a series (x_i), for $i = 1, \ldots, n$, then we define the series of returns by $R_i = \log(x_{i+1}) - \log(x_i)$, for $i = 1, \ldots, n-1$. We can see that: $R_i \approx (x_{i+1} - x_i)/x_i$, that is, it represents an approximation of the change compared to the previous moment. The series of returns appears to be stationary, as can be seen in Figure 3. That is, it represents an approximation of the change compared to the previous moment. By using returns, it is possible to address issues related to trends, heteroscedasticity, and non-stationarity. Due to the differentiation of successive levels, the trend component is eliminated, making the returns series generally more stationary. The returns series appears to be stationary, as seen in Figure 3, and this stationarity facilitates the analysis and modeling of financial data.

Table 3.: Descriptive statistics of stock indices the period from 22 February 2022 to 22 February 2023.

Data set	Minimum	1st quartile	Median	Mean	3rd quartile	Maximum	SD
CAC40	5693	6146	6464	6434	6639	7361	370.9261
AXA SA	20.62	23.31	24.77	24.93	26.74	28.82	2.144032
DAX	11952	13142	13928	13815	14395	15612	852.4276
DVN	50.48	59.57	63.02	64.00	68.95	78.69	6.517895
HANG	14831	19048	20278	19963	21306	23618	1844.771
McDonald	220.2	246.1	254.0	254.8	265.9	281.5	13.05672
NASDAQ	10132	11010	11556	11828	12517	14559	1047.247
Nikkei	24876	26555	27257	27174	27756	29096	811.0432

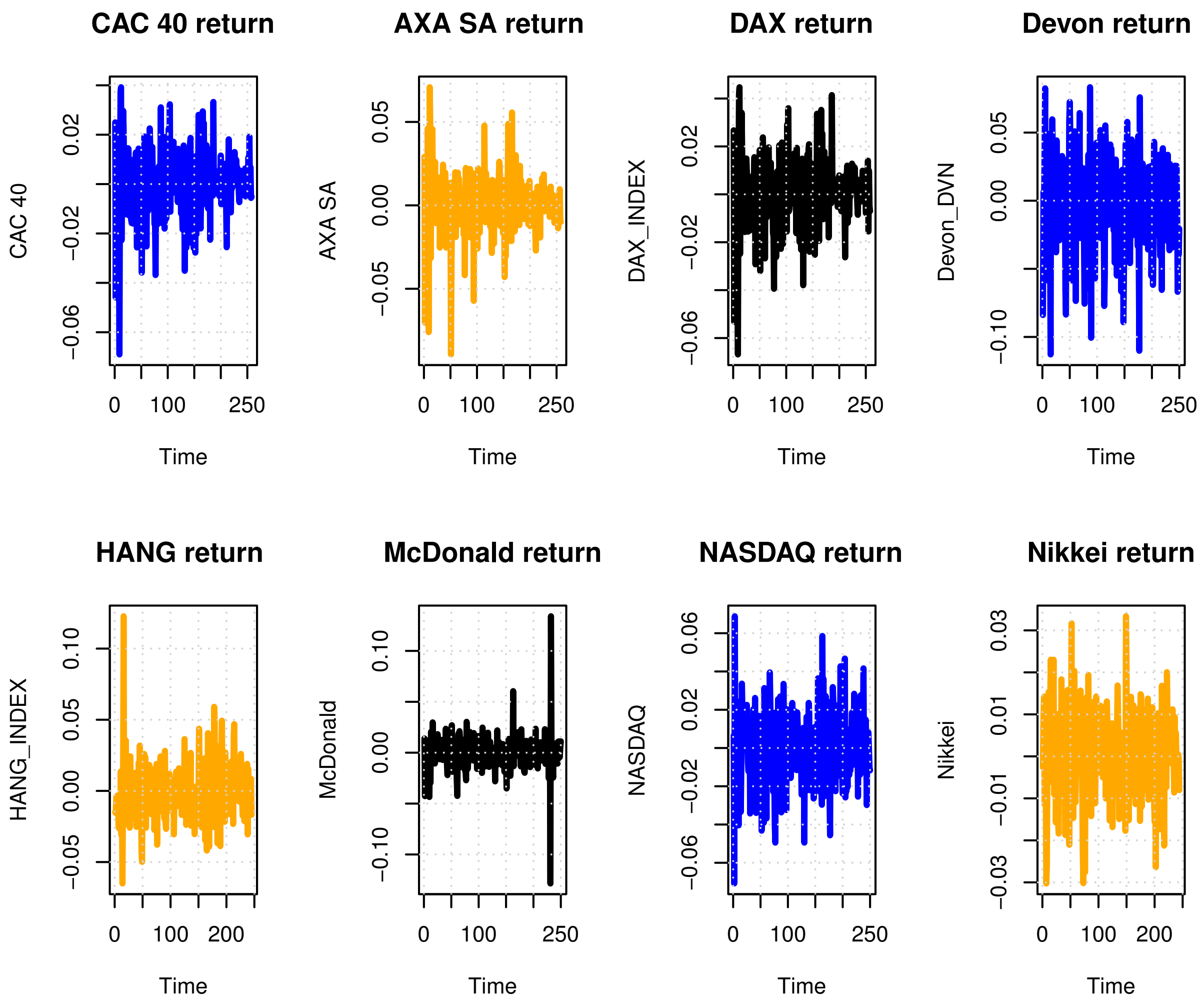

Figure 3.: Return of Stock indices the period from 22 February 2022 to 22 February 2023.

Table 4 illustrates the parameter values of the α-stable distribution for different stock indices using our proposed method and the ML method (see Nolan, 2001). Specifically, the estimated values of the stability index α using the proposed method clearly show that the value of $\hat{\alpha}$ for each stock index is $\hat{\alpha} = 1.95$, except for the CAC 40 ($\hat{\alpha} = 1.871$) and HANG ($\hat{\alpha} = 1.768$). In contrast, the estimated values of α using the ML method are mostly equal to 2. This explains that our proposed method takes into account the asymmetric nature of the data compared to the ML method. This is further confirmed by the estimated values of β; for the proposed method, the values of the skewness parameter β are all different from zero, while those of ML method are mostly equal to zero.

Regarding the scale parameter $\hat{\sigma}$ and σ_{ML}, we observe changes: $\hat{\sigma} > \sigma_{\mathrm{ML}}$ for some stock indices, and $\hat{\sigma} < \sigma_{\mathrm{ML}}$ for others. Similarly, for the location parameter $\hat{\mu}$ and μ_{ML}.

In Tables 3 and 4, by observing the values of: $\hat{\sigma}$ and σ_{ML} from Table 4 with the standard deviation (SD) from Table 3; $\hat{\mu}$ and μ_{ML} from Table 4 with the mean from Table 3; we can see that they are somewhat close. Specifically, if we consider the CAC40 stock index: $\hat{\sigma} = 261.7775$, $\sigma_{\mathrm{ML}} = 258.2216$ and SD $= 370.9261$; $\hat{\mu} = 6445.904$, $\mu_{\mathrm{ML}} = 6434.2911$ and Mean $= 6434$. The four parameters α, β, σ, and μ, uniquely characterize α-stable distributions, indicating that these laws are parametric. It is crucial to understand the statistical significance of each parameter and its impact on the shape of the PDF or CDF.

The parameter α, also known as the characteristic exponent or stability index, characterizes the shape of the distribution or the thickness degree of the distribution tail. The smaller α is, the thicker the distribution tails are, implying a higher probability of observing extreme values. When α is close to two, the probability of observing values far from the centrality is low. A value close to zero of the α index indicates a significant mass in the distribution tail. For example, a Gaussian distribution has the maximum value of α, namely $\alpha = 2$. Smaller α values result in sharper PDF curves, with thicker distribution tails (see Figure 1). The parameter β represents the asymmetry of the distribution, also known as the skewness parameter. When β is equal to -1 (respectively +1), the distribution is entirely skewed to the left (respectively to the right). A positive (respectively negative) β indicates that the mode is to the left (respectively to the right) of the mean. Moreover, a positive β value implies a thicker distribution tail to the right, while a negative β value indicates a thicker distribution tail to the left. β equal to zero corresponds to a symmetric distribution (see Figure 1). The parameter σ is designated as the scale factor. A higher value of σ indicates greater data volatility. By adjusting the σ parameter, one can modulate the width of the distribution body (see Figure 1). The location parameter μ represents, when α is greater than 1, the mean of the distribution law. If $\beta = 0$, then μ is equivalent to the median. In other cases, the μ parameter does not have a direct interpretation (see Figure 1).

Table 4.: Estimated parameters using the proposed method and the ML method with $t_1 = 0.03, t_2 = 0.04, t_3 = 0.09$.

Data set	$\hat{\alpha}$	α_{ML}	$\hat{\beta}$	β_{ML}	$\hat{\sigma}$	σ_{ML}	$\hat{\mu}$	μ_{ML}
CAC40	1.871	2.00	0.846	0.000	258.2216	261.7775	6445.90	6434.29
AXA SA	1.950	2.00	−0.044	0.000	1.8018	1.51313	24.78	24.93
DAX	1.950	2.00	−0.998	0.000	659.6527	601.5927	13928.12	13814.74
DVN	1.950	2.00	0.964	0.000	4.948108	4.599657	63.02	64.00
HANG	1.768	1.50	−1.000	−0.999	1155.779	1029.66	20440.46	20568.455
McDonald	1.95	1.96	−0.994	0.022	10.395	10.38	254.020	254.01
NASDAQ	1.950	1.41	0.868	1.000	791.95	567.2140	11555.97	11413.58
Nikkei	1.950	2.00	0.022	0.000	630.7446	572.3226	27257.35	27174.29

To assess the reliability of these estimators, we plotted the histogram of each stock index along with the PDF using the values estimated by both the proposed method and the ML method, as illustrated in Figure 4. In both cases, the PDF curve manages to model the shape of the data histogram effectively. However, the ML method struggles to adapt to the asymmetric characteristics of the data, whereas the proposed method captures the data well in their nature.

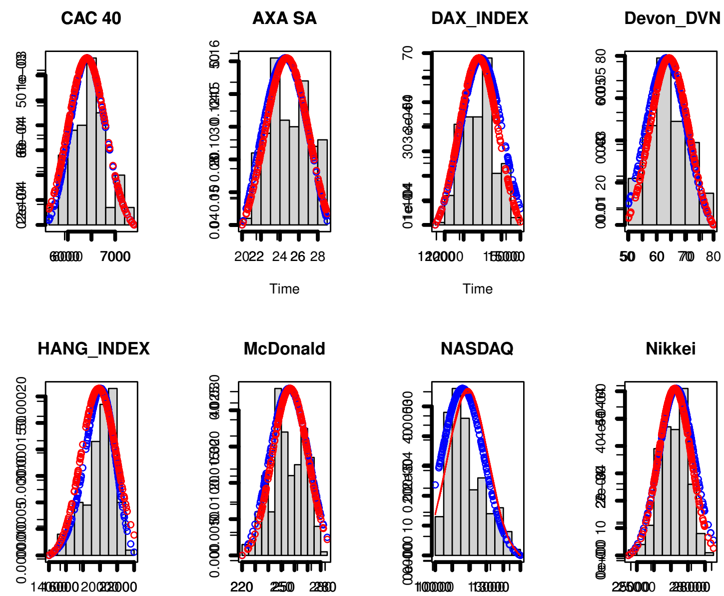

Figure 4.: Proposed method fitting (color blue) versus ML method fitting (color red) of the different stock market indices from 22 February 2022 to 22 February 2023.

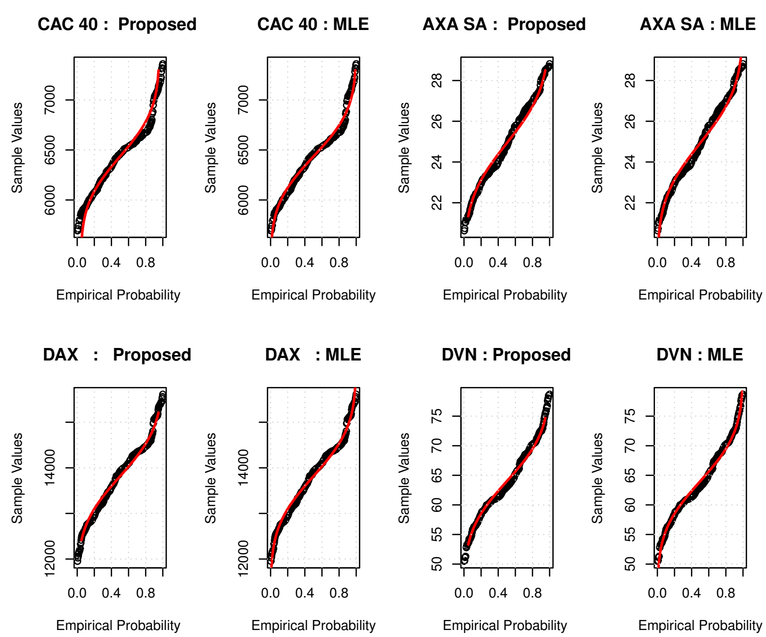

Figure 5.: QQ plot of the proposed method and the ML method fitting from 22 February 2022 to 22 February 2023.

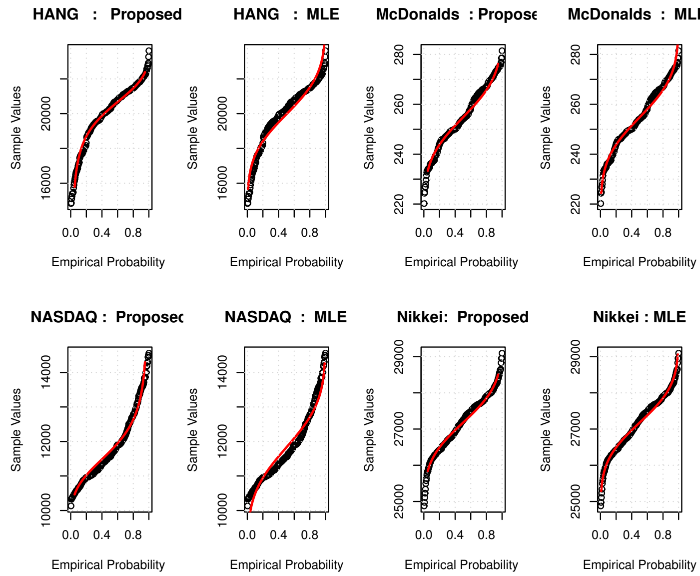

Figure 6.: QQ plot of the proposed method and the ML method fitting from 22 February 2022 to 22 February 2023.

And so to assess the quality of these fits, we plotted the QQ plots for each method. Figures 5 and 6 illustrate these QQ plots, revealing a similarity between the proposed method and the ML method. This similarity can be attributed to the fact that the estimated values of the stability index parameter α by both methods are close. The value at risk (VaR) is a financial risk measure that estimates the potential loss of a portfolio of financial assets at different confidence levels, such as 1%, 5%, and up to 10% in practical cases. It is commonly used in banking regulation to assess necessary reserves. One of the objectives of this study is to compare the VaR estimation obtained from models based on the proposed method and ML method. Theoretical results on this topic are found in reference Mi and Xu (2023). Table 5 provide the VaR obtained from models based on the proposed method and ML method for each stock index and at different confidence levels. With a confidence level of 1%, we observe that the VaR from the proposed method is higher than the VaR from ML method for several stock indices, mostly. At a confidence level of 5%, sometimes the VaR from ML method is higher than the VaR from the proposed method, and sometimes the opposite is true. However, at a confidence level of 10%, the VaR from ML method exceeds that of the proposed method for many stock indices.

Table 5.: VaR of stock market indices.

		Confidence level		
		1%	5%	10%
CAC40	Proposed VaR	−0.07977592	−0.02460576	−0.01460486
	ML VaR	−0.03166673	−0.02228414	−0.01728232
AXA SA	Proposed VaR	−0.07243842	−0.02657139	−0.01682177
	ML VaR	−0.04238662	−0.02987407	−0.02320368
DAX	Proposed VaR	−0.06614101	−0.02449072	−0.0158769
	ML VaR	−0.03388966	−0.02388707	−0.01855473
DVN	Proposed VaR	−0.131893	−0.06455246	−0.04559477
	ML VaR	−0.08128066	−0.05756113	−0.04491633
HANG	Proposed VaR	−0.04448436	−0.0303388	−0.02419092
	ML VaR	−0.04811972	−0.03419181	−0.02676689
McDonald	Proposed VaR	−0.04301373	−0.02273395	−0.01641218
	ML VaR	−0.04096133	−0.02887483	−0.02243155
NASDAQ	Proposed VaR	−0.05321447	−0.03334233	−0.02542094
	ML VaR	−0.0463377	−0.03293035	−0.02578293
Nikkei	Proposed VaR	−0.02441894	−0.0171592	−0.01342032
	ML VaR	−0.02444921	−0.01724275	−0.01340102

6. Application to Moroccan bank credit outstanding balances

To assess the performance of this adjustment with a sample of slightly larger size compared to the samples used in the previous section, we will thus work with a dataset of credit outstanding of size $n = 1000$, which is available on the website of Bank Al Maghrib. This dataset corresponds to the credit outstanding (in millions of dirhams (MA)) of the period between 22 June and 20 March 2023 of Bank Al Maghrib in Morocco (https://www.bkam.ma, accessed on 13 April 2024). We have varied the sample size to observe the impact on each parameter in the model. The credit outstanding balance refers to the total amount of debt that a person or a company owes at a given time. This includes balances on credit cards, personal loans, mortgages, car loans, credit lines, and other forms of credit. In other words, the outstanding balance represents the sum of borrowed amounts that have not yet been repaid, plus interest, fees, and other loan-related charges. Lenders closely monitor their borrowers' outstanding balances to ensure that they can manage their debts and repay them on time. Figure 7 shows the evolution and daily yields of Bank Al Magrib outstanding amounts. The values of the descriptive statistics and the estimated parameter values are listed in Tables 6 and 7, respectively. Figure 8 illustrates the fits with the proposed method and the ML method as well as the `qqplot` of each method.

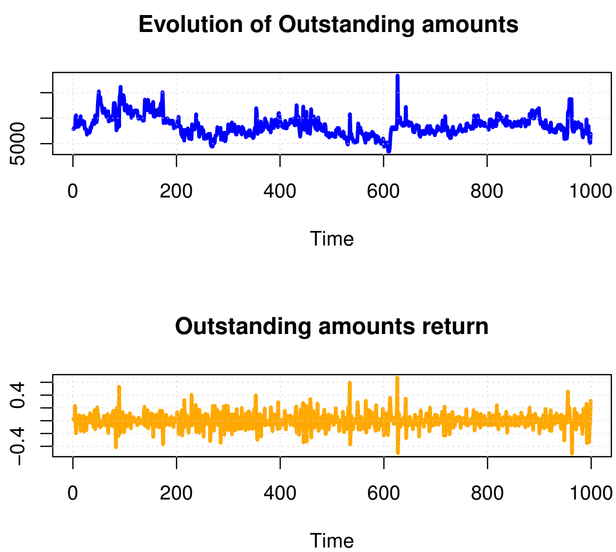

Figure 7.: Evolution and return of amounts from 22 June 2020 to 20 March 2023.

Table 6.: Summary statistical of outstanding amounts from 22 June 2020 to 20 March 2023.

Minimum	1st quartile	Median	Mean	3rd quartile	Max	SD
3465	7276	8321	8499	9447	18364	1916.489

Table 7.: Estimate of parameters using the proposed method and the ML method with $t_1 = 0.03, t_2 = 0.04, t_3 = 0.09$.

Parameters estimate							
$\hat{\alpha}$	α_{ML}	$\hat{\beta}$	β_{ML}	$\hat{\sigma}$	σ_{ML}	$\hat{\mu}$	μ_{ML}
1.6322	1.810	0.56	1.00	1106.321	1214.046	8191.958	8199.753

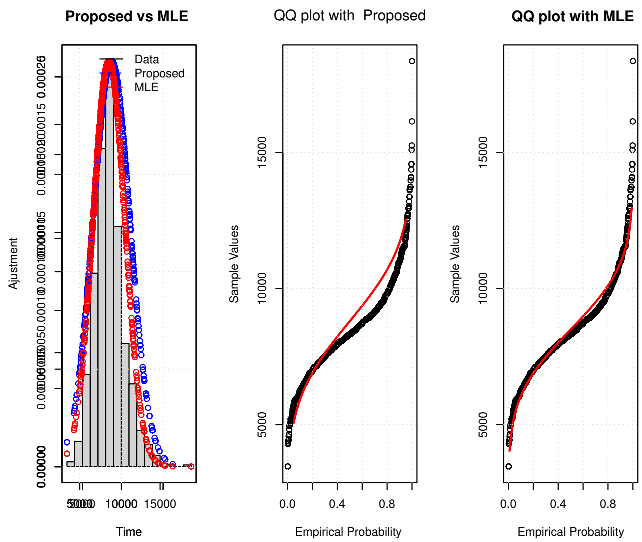

Figure 8.: Proposed method fitting versus ML method fitting of Outstanding amounts June 22, 2020 to March 20, 2023.

7. Conclusions, limitations and future research

Our study has shown that estimating the parameters of an alpha-stable distribution using the characteristic function can be used to model the distribution of returns for various stock indices such as the CAC 40, Nikkei, as well as Bank Al Maghrib outstanding credit. The results have proven that the proposed method using the characteristic function provides a better approximation of the tail distribution of returns than fitting with the maximum likelihood method, which can be helpful for portfolio management, risk management, and asset valuation. The characteristic function has allowed for an efficient estimation of the alpha-stable parameters, which can be applied to financial assets to improve modeling of their distributions. This method can also be used to better capture the extreme events in financial markets that are not typically captured by the maximum likelihood method.

One drawback of our method based on the logarithm of the characteristic function, also known as the cumulant function, is its different behavior, with a much slower convergence rate, not fast, and robustness to small samples, but the advantage lies in its precision in parameter estimation. As for future perspectives, we plan to explore these methods in modeling time series of ARIMA, GARCH, or multiple linear regression type, assuming that the error terms (white noise) follow alpha-stable distributions, and then to observe their predictive behavior. Future work may include applying this method to other financial assets and comparing its results with other methods for estimating tail distributions. Additionally, it would be interesting to investigate the impact of different factors such as volatility and liquidity on the fitting of the alpha-stable distribution. Overall, the findings of this study provide a useful tool for financial analysts and practitioners for modeling the distribution of financial asset returns and managing risk.

In the context of our study, we used the statistical software `R` version `4.4.0` to conduct our analyses. Additionally, we utilized the additional packages `caret` and `ggplot2` for cross-validation and visualization of results, respectively. Regarding the specifications of the computer used, our analyses were performed on a computer equipped with an Intel Core i5 processor and 8 GB of RAM, 500GB SSD hard disk. Execution times varied depending on the size of the dataset and the complexity of the models. The real dataset used in this work is available online and free of charge. The `R` codes for the simulation study are available from the author on reasonable request.

Appendix A: Proof of Theorem 1

Proof . For $\alpha \neq 1$, the characteristic function of the random variable X is given by

$$\log \Phi_X(t) = \mathrm{i}\mu t - \sigma^\alpha |t|^\alpha \left(1 - \mathrm{i}\beta \mathrm{sign}(t) \tan\left(\frac{\pi\alpha}{2}\right)\right) = -\sigma^\alpha |t|^\alpha + \mathrm{i}\left[\sigma^\alpha |t|^\alpha \beta \mathrm{sign}(t) \tan(\frac{\pi\alpha}{2}) + \mu t\right].$$

For $t \in \mathrm{R}_+^*$, we have

$$\log(\Phi_X(t)) = -\sigma^\alpha t^\alpha + \mathrm{i}\left[\sigma^\alpha t^\alpha \beta \tan\left(\frac{\pi\alpha}{2}\right) + \mu t\right].$$

We set

$$\Re(\log(\tilde{\Phi}_X(t_j))) = -\sigma^\alpha t_j^\alpha, \quad j = 1, 2, \tag{7.11}$$

and

$$\Im(\log(\tilde{\Phi}_X(t_j))) = \sigma^\alpha t_j^\alpha \beta \tan\left(\frac{\pi\alpha}{2}\right) + \mu t_j, \quad j = 1, 2. \tag{7.12}$$

We check the results in the case where $n = 2$. From Equation (7.11), we find that

$$\sum_{j=1}^{2} \log\left(-\Re(\log(\tilde{\Phi}_X(t_j)))\right) = \alpha \sum_{j=1}^{2} (\log(\sigma) + \log(t_j)).$$

Therefore, we have that

$$\alpha = \frac{\sum_{j=1}^{2} \log(-\Re(\log(\tilde{\Phi}_X(t_j))))}{\sum_{j=1}^{2} (\log(\sigma) + \log(t_j))} = \frac{A}{2\log(\sigma) + B}.$$

Now, we replace α by its expression in Equation (7.11) for $j = 1$ or 2. Thus, for $j = 1$ in Equation (7.11), we find that

$$\log(\sigma) = \frac{B \log\big(-\Re(\log(\tilde{\Phi}_X(t_1)))\big) - A\log(t_1)}{A - n\log\big(-\Re\big(\log\big(\tilde{\Phi}_X(t_1)\big)\big)\big)}.$$

From Equation (7.12), we find the relationship stated as

$$\sum_{j=1}^{2} \Im\big(\log(\tilde{\Phi}_X(t_j))\big) = \beta\sigma^\alpha \tan\left(\frac{\pi\alpha}{2}\right) \sum_{j=1}^{2} t_j^\alpha + \mu \sum_{j=1}^{2} t_j. \tag{7.13}$$

We deduce β through Equation (7.13). Thus, we find μ by replacing β by its expression given in Equation (7.12) for $j = 1$ or 2. The cases where $j = 3, 4, 5 \ldots$ are done in the same way. ∎

Appendix B: Proof of Theorem 2

Proof For $\alpha = 1$, the characteristic function of the standard Zolotarev (1998) parametrization is given by

$$\log(\Phi_X(t)) = \mathrm{i}\mu t - \sigma|t|\Big(1 + \mathrm{i}\beta \mathrm{sign}(t)\frac{2}{\pi}\log(|t|)\Big) = -\sigma|t| + \mathrm{i}[\mu t - \sigma|t|\beta \mathrm{sign}(t)\frac{2}{\pi}\log(|t|)]$$
$$= -\sigma t + \mathrm{i}[\mu t - \sigma t\beta\frac{2}{\pi}\log(t)], \quad \forall t \in \mathrm{R}_+^*.$$

We set

$$\Re\big(\log(\tilde{\Phi}_X(t_j))\big) = -\sigma t_j, \quad \forall t_j \in \mathrm{R}_+^*, \quad j = 1, 2, \tag{7.14}$$

and

$$\Im\big(\log(\tilde{\Phi}_X(t_j))\big) = \mu t_j - \sigma t_j \beta\frac{2}{\pi}\log(t_j), \quad \forall t_j \in \mathrm{R}_+^*, j = 1, 2. \tag{7.15}$$

From Equation (7.14), for $j = 1, 2$, we find that

$$\sum_{j=1}^{2}\Re\big(\log(\tilde{\Phi}_X(t_j))\big) = -\sigma\sum_{j=1}^{2} t_j. \tag{7.16}$$

Therefore, we obtain σ through Equation (7.16). By using Equation (7.15), for $j = 1, 2$, we obtain the relation presented as

$$\sum_{j=1}^{2}\Im\left(\log(\tilde{\Phi}_X(t_j))\right) = \mu\sum_{j=1}^{2} t_j - \sigma\beta\frac{2}{\pi}\sum_{j=1}^{2} t_j\log(t_j). \tag{7.17}$$

Through Equation (7.17), we reach an expression for μ. Then, it is enough to replace μ in Equation (7.15), with $j = 1, 2$, to get the asymmetry parameter β. It is easy to check that the result remains true for $j = 3, 4, 5, \ldots$ ■

Author contributions Conceptualization: C.B.D., C.G.; data curation: C.B.D.; formal analysis: C.B.D., E.K.M.; methodology: C.B.D., C.G.; software: C.B.D., C.G.; supervision: E.K.M.; validation: C.B.D., C.G.; writing —original draft: C.B.D.; writing —review and editing: C.B.D., C.G. All authors have read and agreed to the published version of the article.

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