CHILEAN
JOURNAL OF
STATISTICS

Edited by Víctor Leiva and Carolina Marchant

A free open-access journal indexed by

EMERGING SOURCES CITATION (ISI WEB OF SCIENCE) — INDEXED BY — CLARIVATE ANALYTICS

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Volume 13 Number 1
April 2022
ISSN: 0718-7912 (print)
ISSN: 0718-7920 (online)

Published by the
Chilean Statistical Society
SOCHE
SOCIEDAD CHILENA DE ESTADÍSTICA

Aims

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Volume 13, Number 1 (April 2022), Pages 91–111
DOI: 10.32372/ChJS.13-01-05

Distribution Theory
Research Paper

The Mc-Donald Chen distribution: A new bimodal distribution with properties and applications

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(Received: 24 June 2021 · Accepted in final form: 02 March 2022)

Abstract

In this paper, the McDonald-Chen distribution is proposed and studied to model different type of data. Its probability density function allows bimodality, thus showing that the model is very flexible. Its failure or hazard rate function may have increasing, decreasing, bathtub, inverted bathtub and increasing-decreasing-increasing shapes depending on the parameter values. The new distribution includes at least five major special cases. Some of its mathematical properties are addressed. The maximum likelihood method is adopted to estimate the model parameters. Monte Carlo simulations evaluate the accuracy of the maximum likelihood estimators. The new distribution is better than three other popular distributions to model two real data sets.

Keywords: Chen distribution · Family of distributions · Maximum likelihood method · Moments · Monte Carlo Simulation.

Mathematics Subject Classification: 46N30 · 78M31.

1. Introduction

Several distributions have been proposed to model data in real applications. Lai (2013) detailed the importance of building new survival distributions and the fact that the failure or hazard rate curves could accommodate different shapes. Thus, there is a need for distributions that are quite flexible to model these shapes. Among the different mechanisms for proposing new continuous distributions, we have: transformation of the random variable; random variable convolution; random variable composition (Cordeiro et al., 2018); mixing distributions between random variables (Nedjar and Zeghdoudi, 2016); distributions that transform the cumulative distribution (Bourguignon et al., 2014). The choice of generated distributions can be carried out using transformation in the cumulative distribution. Some distributions generated using this technique are the beta modified Weibull (Silva et al., 2010), gamma modified Weibull (Cordeiro et al., 2015), transmuted Dagum (Elbatal and Aryal, 2015), Harris extended Lindley (Cordeiro et al., 2019).

Eugene et al. (2002) pioneered the beta-generalized (beta-G) family, which includes nearly all of well-known models as special cases. Further, it can give lighter and heavier tails and be

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ISSN: 0718-7912 (print)/ISSN 0718-7920 (online)

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applied in several areas such as engineering and biological research, among others. Explicit expressions are reported in several published papers, which facilitate to find its mathematical properties for special models. In the last years, several beta-G models have been proposed; see the list of forty five special models in Table 3 of Tahir et al. (2015). This family has the major benefit for fitting skewed data that can not be fitted by most well-known continuous distributions.

In this paper, a flexible extension of the Chen distribution (Chen, 2000) is proposed, which can be useful in several practical contexts. In particular, adding shape parameters to a baseline distribution can provide better fits to real data in different settings and extended Chen distribution has interesting mathematical properties.

The paper is unfolded as follows. In Section 2, a brief introduction to the McDonald-Chen (MC) distribution is given. In Section 3, the quantile function (QF) of the MC distribution is determined. In Section 4, the new probability density function (PDF) is expressed as a linear combination of Chen PDFs. Moments and moments generating function are obtained in Section 5. In Section 6, its parameters are estimated by the maximum likelihood (ML) method. In Section 7, some simulation results verify the precision of the parameter estimates. In Section 8, the MC distribution is proved to outperform some well-known lifetime models. Finally, Section 9 offers some concluding remarks.

2. Background

Based on the beta-G family, Alexander et al. (2012) defined the cumulative distribution function (CDF) and PDF of the McDonald-generalized (MG) class of distributions as

$$F(x;a,b,c,\boldsymbol{\theta}) = \frac{B_{G(x;\boldsymbol{\theta})^c}(a,b)}{B(a,b)} = \frac{1}{B(a,b)}\int_0^{G(x)^c} w^{a-1}(1-w)^{b-1}\mathrm{d}w \tag{1}$$

and

$$f(x;a,b,c,\boldsymbol{\theta}) = \frac{c}{B(a,b)} g(x;\boldsymbol{\theta})\, G(x;\boldsymbol{\theta})^{ac-1}\,[1-G(x;\boldsymbol{\theta})^c]^{b-1}, \tag{2}$$

respectively, where $\boldsymbol{\theta}$ is the parameter vector of the baseline distribution $G(x;\boldsymbol{\theta})$, $g(x;\boldsymbol{\theta}) = \mathrm{d}(x;\boldsymbol{\theta})/\mathrm{d}x$, a, b and c are three positive additional shape parameters, $B(a,b) = \int_0^1 w^{a-1}(1-w)^{b-1}\mathrm{d}w$ denotes the beta function and $B_z(a,b) = \int_0^z w^{a-1}(1-w)^{b-1}\mathrm{d}w$ denotes the lower incomplete beta function.

Let $X \sim \mathrm{MG}(a,b,c,\boldsymbol{\theta})$ be a random variable X having PDF as given in Equation (2). Although this transformation is simple, the MG family is richer than the corresponding baseline $G(x)$. For $G(x) = x$, the MG family reduces to the McDonald distribution pioneered by Mcdonald (2008). For $c = 1$ in Equation (1), it follows the beta-G class defined by Eugene et al. (2002). For $a = 1$, Equation (1) coincides with the Kumaraswamy-generalized (Kw-G) class introduced by Cordeiro and de Castro (2011). The MG family is quite important, since it includes as special cases two of the most well-known classes in the literature, which generated many published distributions in the last twenty years. According to Cordeiro et al. (2012a), the MG family allows greater flexibility in its tails and can be widely used in engineering, biology and other areas.

The hazard rate function (HRF) of X is given by

$$\tau(x;a,b,c,\boldsymbol{\theta}) = \frac{cg(x;\boldsymbol{\theta})G(x;\boldsymbol{\theta})^{ac-1}[1-G(x;\boldsymbol{\theta})^c]^{b-1}}{1-B_{G(x;\boldsymbol{\theta})^c}(a,b)}. \tag{3}$$

The Chen distribution is taken as baseline, since it allows us to model data with bathtub HRF. The CDF and PDF of the Chen distribution are stated as

$$G(y;\lambda,\beta) = 1 - \mathrm{e}^{\lambda(1-\mathrm{e}^{y^{\beta}})},\ y > 0 \tag{4}$$

and

$$g(y;\lambda,\beta) = \lambda\beta y^{\beta-1}\mathrm{e}^{y^{\beta}+\lambda(1-\mathrm{e}^{y^{\beta}})},\ y > 0, \tag{5}$$

respectively, where $\lambda > 0$ is the scale parameter and $\beta > 0$ is the shape parameter. Henceforth, $Y \sim \mathrm{Chen}(\lambda,\beta)$ denotes a random variable with PDF as given in Equation (5).

By taking G and g as the CDF and PDF of the Chen distribution, respectively, and substituting in Equations (1), (2) and (3), the CDF, PDF and HRF of the MC distribution are formulated as

$$F(x;a,b,c,\lambda,\beta) = \frac{1}{B(a,b)}\int_0^{\left[1-\mathrm{e}^{\lambda(1-\mathrm{e}^{x^{\beta}})}\right]^c} w^{a-1}(1-w)^{b-1}\mathrm{d}w, \tag{6}$$

$$f(x;a,b,c,\lambda,\beta) = \frac{c\lambda\beta}{B(a,b)}x^{\beta-1}\mathrm{e}^{x^{\beta}+\lambda(1-\mathrm{e}^{x^{\beta}})}\left[1-\mathrm{e}^{\lambda(1-\mathrm{e}^{x^{\beta}})}\right]^{ac-1}\left\{1-\left[1-\mathrm{e}^{\lambda(1-\mathrm{e}^{x^{\beta}})}\right]^c\right\}^{b-1} \tag{7}$$

and

$$\tau(x;a,b,c,\lambda,\beta) = \frac{c\lambda\beta x^{\beta-1}\mathrm{e}^{x^{\beta}+\lambda(1-\mathrm{e}^{x^{\beta}})}\left[1-\mathrm{e}^{\lambda(1-\mathrm{e}^{x^{\beta}})}\right]^{ac-1}\left\{1-\left[1-\mathrm{e}^{\lambda(1-\mathrm{e}^{x^{\beta}})}\right]^c\right\}^{b-1}}{1-\int_0^{\left[1-\mathrm{e}^{\lambda(1-\mathrm{e}^{x^{\beta}})}\right]^c} w^{a-1}(1-w)^{b-1}\mathrm{d}w},$$

respectively.

Henceforth, let $X \sim \mathrm{MC}(a,b,c,\lambda,\beta)$ have PDF as given in Equation (7). For $c = 1$, the MC distribution becomes the beta-Chen (BC), not yet known in the literature. For $a = 1$, it follows the (new) Kumaraswamy-Chen (KC). Further, Equation (7) reduces to the exponentiated-Chen ($b = c = 1$) (Dey et al., 2017), exponentiated-Chen Lehmann type 2 ($a = c = 1$) and the Chen itself ($a = b = c = 1$) distributions.

Figure 1 displays plots of the MC PDFs for selected parameter values, where it is shown that this distribution is quite flexible having several forms including bimodality.

The HRF curves for some parameter choices are given in Figure 2. The HRF of X can be increasing, decreasing, unimodal, crescent-descending-crescent and bathtub shape, which shows once again its great flexibility.

Summing up what was said above, we cite six basic motivations for the MC distribution: (i) greater flexibility in the PDF and HRF. In fact, its PDF has bimodality, increasing, decreasing, bathtub and inverted shapes of the HRF, whereas the Chen PDF has only increasing, decreasing and unimodal shapes. In addition, the rate of the MC distribution can be increasing, decreasing, bathtub, inverted bathtub, and increasing-decreasing-increasing shapes. This last form exists for few distributions, but it can be found in many real data sets; (ii) make the skewness and kurtosis more flexible compared to the Chen distribution. The parameter c of the new distribution changes substantially the values of its skewness and kurtosis as shown in the plots of Figures 3 and 4, thus making it very interesting for real applications; (iii) provide consistently better fits than other lifetime models as proved empirically in Section 8; (iv) the proposed distribution includes five others sub-models that

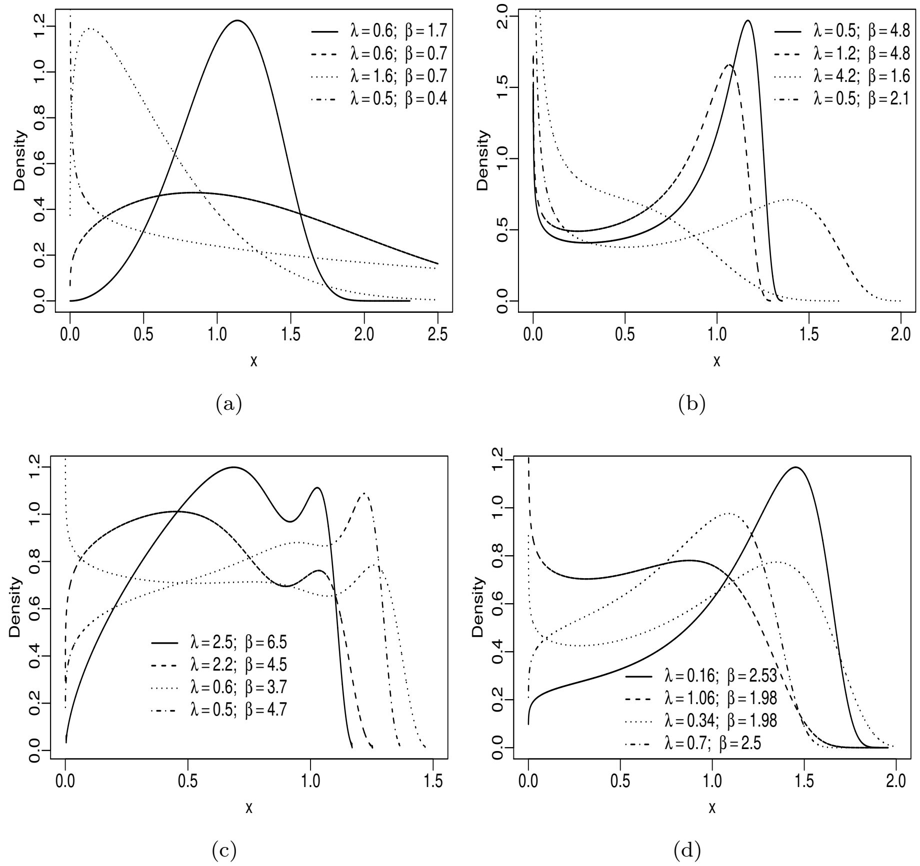


Figure 1. The MC PDF for some parameters values: (a) $\mathrm{MC}(0.5, 0.8, 3.5, \lambda, \beta)$, (b) $\mathrm{MC}(0.86, 0.32, 0.18, \lambda, \beta)$, (c) $\mathrm{MC}(0.05, 0.2, 5, \lambda, \beta)$ and (d) $\mathrm{MC}(0.5, 0.5, 0.9, \lambda, \beta)$.

can be compared by using likelihood ratio (LR) tests to choose the best model to explain a data set; (v) the properties of the new distribution are easily obtained from those of Chen due to a linear representation for its PDF; and (vi) construct heavy–tailed special cases that are not longer-tailed for modeling real data.

3. Quantile function

The QF of the MG family, say $Q(u; a, b, c, \boldsymbol{\theta}) = F^{-1}(u; a, b, c, \boldsymbol{\theta})$, can be expressed in terms of the beta QF. Basically, according to Cordeiro et al. (2012b), the QF of the MG distribution (for $0 < u < 1$) has the form

$$Q(u; a, b, c, \boldsymbol{\theta}) = Q_{\mathrm{G}}\{Q_{\beta}(u; a, b)^{\frac{1}{c}}; \boldsymbol{\theta}\},$$

where Q_{G} is the QF of the baseline G and $Q_{\beta}(u; a, b)$ is the beta QF with parameters a and b; see the Wolfram website at `http://functions.wolfram.com/06.23.06.0004.01`.

Thus, the QF of the MC distribution can be expressed as

$$Q(u; a, b, c, \lambda, \beta) = \left\{ \log \left[1 - \frac{1}{\lambda} \log \left(1 - Q_{\beta}(u; a, b)^{\frac{1}{c}} \right) \right] \right\}^{\frac{1}{\beta}}, \ 0 < u < 1.$$

The simulation of X is very easy. If U is a uniform random variable on the unit interval

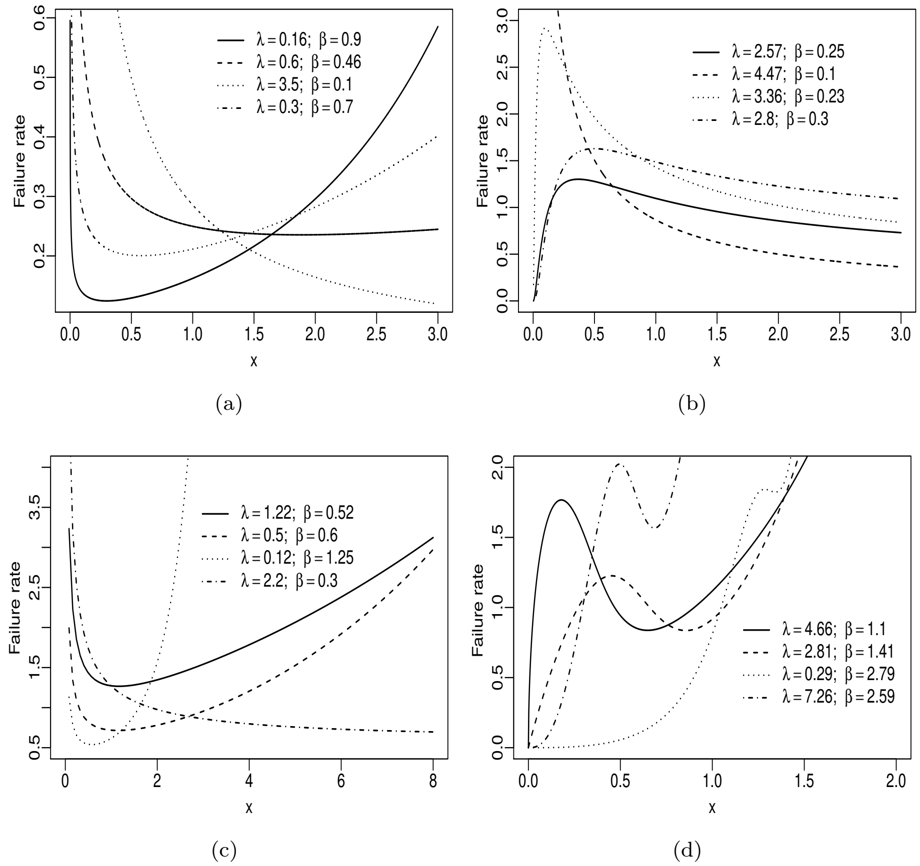


Figure 2. The HRF of the MC model for some parameters values: (a) $\mathrm{MC}(0.6, 0.3, 1.35, \lambda, \beta)$, (b) $\mathrm{MC}(1.2, 0.7, 15, \lambda, \beta)$, (c) $\mathrm{MC}(0.66, 0.7, 0.35, \lambda, \beta)$ and (d) $\mathrm{MC}(0.07, 0.08, 20, \lambda, \beta)$.

$(0, 1)$, then

$$X = \left\{ \log \left[1 - \frac{1}{\lambda} \log \left(1 - Q_\beta(U; a, b)^{\frac{1}{c}} \right) \right] \right\}^{\frac{1}{\beta}},$$

wis an MC distributed random variable.

Let $Q(u) = Q(u; a, b, c, \lambda, \beta)$ be the QF of the MC distribution by omitting the arguments. The baseline parameters are $\lambda = 5$ and $\beta = 1.62$ and c varies in $\{0.2, 1, 5, 10\}$ for the scenarios (a)-(d), respectively, to study the influence of the generator parameters a and b on the skewness and kurtosis of the MC distribution. The parameters a and b vary in the interval $(0.1, 1)$. Figure 3 displays the Bowley skewness, as functions of a and b, defined as

$$B = \frac{Q(3/4) + Q(1/4) - 2\,Q(2/4)}{Q(3/4) - Q(1/4)}.$$

The minimum and maximum values for B are then $(-0.1542,\ 1.0000)$, $(-0.1272,\ 0.8406)$, $(-0.0690,\ 0.3101)$ and $(-0.0425,\ 0.2649)$ for the scenarios (a)-(d), respectively. For the selected parameter values, the asymmetry becomes increasingly negative when c increases.

Consider the same parameter values for the Moor kurtosis, as functions of a and b, expressed as

$$M = \frac{Q(7/8) - Q(5/8) - Q(3/8) + Q(1/8)}{Q(6/8) - Q(2/8)},$$

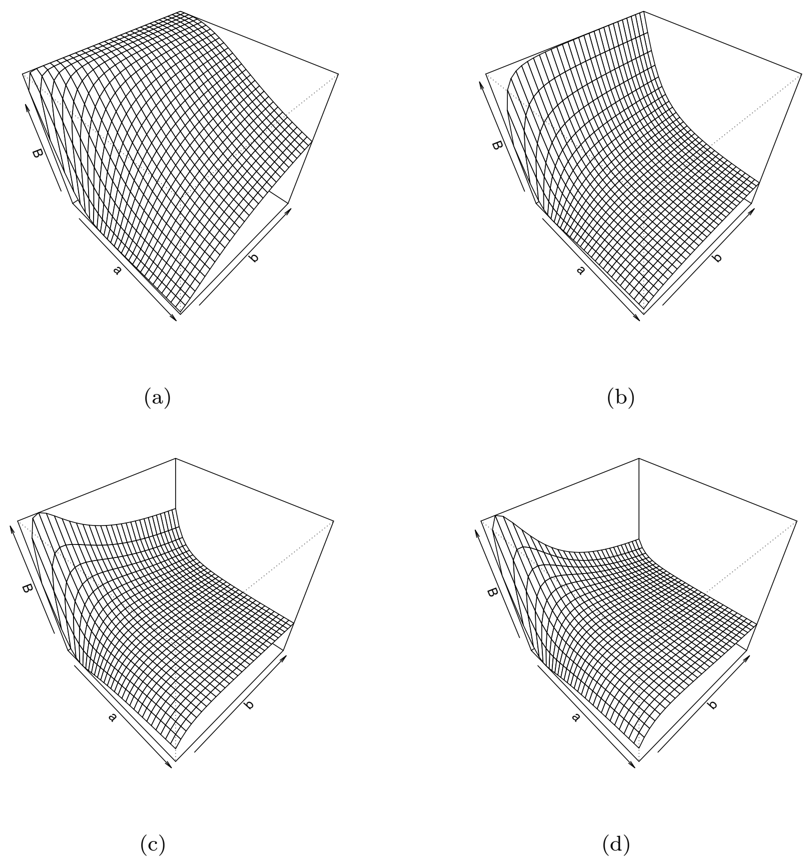


Figure 3. Bowley skewness as function of c: (a) $c = 0.2$, (b) $c = 1$, (c) $c = 5$ and (d) $c = 10$.

Figure 4 displays the Moor kurtosis, where the minimum and maximum values of M are $(-0.1985, 116.5147)$, $(-0.1667, 2.4161)$, $(-0.0744, 0.6191)$ and $(-0.0260, 0.4811)$ for the scenarios (a)-(d), respectively. Small values of c give higher kurtosis. The kurtosis decreases and stabilizes when c increases.

4. Linear representation

Equations (6) and (7) can be expressed in terms of exponentiated distributions. For a given CDF $G(z;\boldsymbol{\theta})$ with parameter vector $\boldsymbol{\theta}$, the random variable Z is exponentiated-G (exp-G) distributed, with power parameter $a > 0$, if its CDF and PDF are

$$H(z;a,\boldsymbol{\theta}) = G(z;\boldsymbol{\theta})^{a}, \quad h(x) = a\, g(z;\boldsymbol{\theta})G(z;\boldsymbol{\theta})^{a-1},$$

respectively, where $g(z;\boldsymbol{\theta}) = \mathrm{d}G(z;\boldsymbol{\theta})/\mathrm{d}z$. The exp-G model is also called the Lehmann type I distribution. From now on, we denote it as $Z \sim \text{exp-G}(a,\boldsymbol{\theta})$.

Following Alexander et al. (2012), Equation (2) can be expressed as

$$f(x;a,b,c,\boldsymbol{\theta}) = \sum_{k=0}^{\infty} b_k\, h(x;c(a+k),\boldsymbol{\theta}), \tag{8}$$

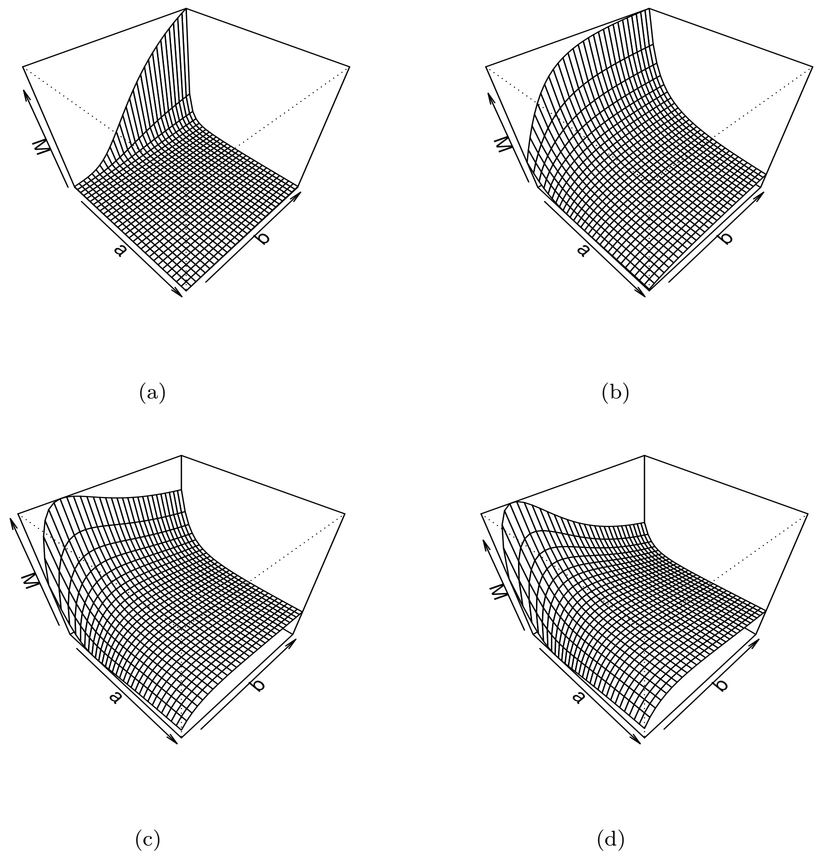


Figure 4. Moor kurtosis as function of c: (a) $c = 0.2$, (b) $c = 1$, (c) $c = 5$ and (d) $c = 10$.

where $h(x; c(a + k), \boldsymbol{\theta})$ is the exp-G($c(a + k), \boldsymbol{\theta}$) PDF, and the coefficients b_k are

$$b_k = \frac{(-1)^k \, \Gamma(a + b)}{(a + k) \, k! \, \Gamma(a) \, \Gamma(b - k)},$$

where $\Gamma(p) = \int_0^\infty w^{p-1} \mathrm{e}^{-w} \mathrm{d}w$ denotes the gamma function. We can prove that $\sum_{k=0}^\infty b_k = 1$.

Equation (8) reveals that MG PDF is a linear combination of exp-G PDFs. Thus, several MG properties can be determined by knowing those corresponding exp-G properties (Cordeiro et al., 2012a). By integrating Equation (8), the MG CDF follows as

$$F(x; a, b, c, \boldsymbol{\theta}) = \sum_{k=0}^{\infty} b_k \, H(x; c(a + k), \boldsymbol{\theta}),$$

where $H(x; c(a + k), \boldsymbol{\theta})$ is the exp-G($c(a + k), \boldsymbol{\theta}$) CDF.

Theorem 4.1 Let Y be a random variable having a Chen CDF as given in Equation (4). Then, the CDF and PDF of the exp-Chen(a, λ, β) distribution are stated as

$$H(y; a, \lambda, \beta) = 1 + \sum_{m=1}^{\infty} (-1)^m \binom{a}{m} [1 - G(y; m\lambda, \beta)]$$

and

$$h(y;a,\lambda,\beta)=\sum_{m=1}^{\infty}w_m(a)\,g(y;m\lambda,\beta),$$

respectively, where $w_m(a)=(-1)^{m+1}\binom{a}{m}$.

Proof For $|x|<1$ and any real $a\neq 0$, the convergent power series holds by means of

$$(1-x)^a=\sum_{m=0}^{\infty}(-1)^m\binom{a}{m}x^m.$$

Thus, the CDF of the exp-Chen distribution is given by

$$H(y;a,\lambda,\beta)=\left[1-\mathrm{e}^{\lambda(1-\mathrm{e}^{y^\beta})}\right]^a=\sum_{m=0}^{\infty}(-1)^m\binom{a}{m}\mathrm{e}^{m\lambda(1-\mathrm{e}^{y^\beta})}$$
$$=1+\sum_{m=1}^{\infty}(-1)^m\binom{a}{m}[1-G(y;m\lambda,\beta)].$$

By differentiating the last equation, we have that

$$h(y;a,\lambda,\beta)=\sum_{m=1}^{\infty}(-1)^{m+1}\binom{a}{m}g(y;m\lambda,\beta),$$

which shows that the exp-Chen PDF is a linear combination of Chen PDFs. ■

Based on Equation (8) and Theorem 4.1, the PDF of X can be expressed as

$$f(x;a,b,c,\lambda,\beta)=\sum_{m=1}^{\infty}d_m\,g(x;m\lambda,\beta), \tag{9}$$

where

$$d_m=d_m(a,b,c)=\sum_{k=0}^{\infty}\frac{(-1)^{k+m+1}\Gamma(a+b)}{(a+k)\,k!\,\Gamma(a)\,\Gamma(b-k)}\binom{c(a+k)}{m},$$

and $g(x;m\lambda,\beta)$ is the Chen PDF with scale parameter $m\lambda$ and shape parameter β. Clearly, the shape parameters of the MC generation are restricted to the coefficients in Equation (9).

Some mathematical properties of the MC distribution can be derived from Equation (9) and those properties of the Chen distribution. For example, the ordinary and incomplete moments and moment generating function (MGF) of X can be determined from the corresponding quantities of the Chen distribution. Consequently, the beta-Chen and Kw-Chen PDFs are also linear combinations of Chen PDFs when $c=1$ and $a=1$, respectively.

By integrating Equation (9), the CDF of the MC distribution is given by

$$F(x;a,b,c,\lambda,\beta)=\sum_{m=1}^{\infty}d_m\,G(x;m\lambda,\beta),$$

where $G(x;m\lambda,\beta)$ is the CDF of the Chen$(m\lambda,\beta)$ distribution.

5. Moments and Moment generating function

Let Y_m be a random variable having the Chen PDF with scale parameter $m\lambda$ and shape parameter β, that is, $Y_m \sim \text{Chen}(m\lambda, \beta)$. By using Equation (9), the rth moment of X can be written as

$$\mathrm{E}[X^r] = \sum_{m=1}^{\infty} d_m\, \mathrm{E}[Y_m^r].$$

Pogany et al. (2017) demonstrated that the rth moment of Y has the form

$$\mathrm{E}[Y^r] = \lambda\, \mathrm{e}^{\lambda}\, \mathrm{D}_t^{r\beta^{-1}} \left[\frac{\Gamma(t+1,\lambda)}{\lambda^{t+1}}\right]_{t=0}. \tag{10}$$

Here, we have that

$$\mathrm{D}_t^{p} \left[\frac{\Gamma(t+1,\lambda)}{\lambda^{t+1}}\right]_{t=0} = \Gamma(p+1) \sum_{k\geq 0} \frac{(2)_k}{k!}\, \Phi_{\mu,1}^{(0,1)}(-k, p+1, 1)\; {}_1F_1(k+2; 2; -\lambda),$$

where $\Phi_{\mu,1}^{(0,1)}(-a, p+1, 1) = \sum_{n\geq 0} (-a)^n / n!(n+1)^{p+1}$ for $\mu \in \mathrm{C}$, ${}_1F_1(a; b; x) = \sum_{n\geq 0} (a)_n x^n/(b)_n n!$, for $x, a \in \mathrm{C}$ and $b \in \mathrm{C} \setminus Z_0^-$, is the confluent hypergeometric function (Kilbas et al., 2006, p. 29, Eq. 1.6.14) and $(\lambda)_\eta = \Gamma(\lambda+\eta)/\Gamma(\lambda)$, for $\lambda \in \mathrm{C} \setminus \{0\}$, is the generalized Pochhammer symbol, under the convention $(0)_0 = 1$.

The rth ordinary moment of X follows from Equation (10) as

$$\mathrm{E}[X^r] = \lambda \sum_{m=1}^{\infty} m\, d_m\, \mathrm{e}^{m\lambda}\, \mathrm{D}_t^{r\beta^{-1}} \left[\frac{\Gamma(t+1, m\lambda)}{(m\lambda)^{t+1}}\right]_{t=0}.$$

The incomplete moments of a distribution have great applicability to measure inequality. The first incomplete moment is used to construct Lorenz and Bonferroni curves.

For $z > 0$, the rth incomplete moment of Y, say $q_r(z; \lambda, \beta) = \int_0^z y^r\, g(y; \lambda, \beta)\mathrm{d}y$, follows from Pogany et al. (2017) as

$$q_r(z; \lambda, \beta) = \lambda \mathrm{e}^{\lambda} \sum_{n,k\geq 0} \sum_{j=1}^{k} \frac{(2)_{n+k}}{(2)_n} \frac{(-1)^{n+j} \lambda^n \binom{k}{j}}{n!k!(j+1)^{r\beta^{-1}+1}} \gamma(r\beta^{-1}, (j+1)(1-z^{-1})), \tag{11}$$

where $\gamma(p, z) = \int_0^z w^{p-1}\, \mathrm{e}^{-w}\mathrm{d}w$ denotes the lower incomplete gamma function.

The rth incomplete moment of X can be expressed from Equation (9) as

$$m_r(z) = \sum_{m=1}^{\infty} d_m\, q_r(z; m\lambda, \beta),$$

which depends directly on the rth incomplete moment of the $\text{Chen}(m\lambda, \beta)$ distribution.

By using Equation (11), the rth incomplete moment de X can be written as

$$m_r(z) = \lambda \sum_{m=1}^{\infty} m\, \mathrm{e}^{m\lambda}\, d_m \sum_{n,k\geq 0} \sum_{j=1}^{k} \frac{(2)_{n+k}}{(2)_n} \frac{(-1)^{n+j} (m\lambda)^n \binom{k}{j}}{n!k!(j+1)^{r\beta^{-1}+1}} \gamma(r\beta^{-1}, (1-z^{-1})(j+1)).$$

The MGF of Y, say $M_Y(t) = \mathrm{E}\left[\mathrm{e}^{-tY}\right]$, for $t > 0$, can be determined from Pogany et al. (2017) as

$$M_Y(t) = \lambda\,\beta\,\mathrm{e}^{\lambda}\,t^{-\beta}\sum_{n\geq 0}\frac{(-\lambda)^n}{n!}\,{}_1\Psi_0\left[(\beta,\beta);-;\frac{n+1}{t^{\beta}}\right], \tag{12}$$

where

$${}_1\Psi_0\left[(a,b);-;z\right] = \sum_{n\geq 0}\frac{\Gamma(a+bn)\,z^n}{n!}, \qquad z, a \in \mathrm{C}, b > 0,$$

is the generalized Fox-Wright function. Thus, using Equations (9) and (12), the MGF of X is stated as

$$M_X(t) = \lambda\,\beta\,\mathrm{e}^{\lambda}\,t^{-\beta}\sum_{m=1}^{\infty}\sum_{n\geq 0}\frac{(-m\lambda)^n\,d_m}{n!}\,{}_1\Psi_0\left[(\beta,\beta);-;\frac{n+1}{t^{\beta}}\right].$$

6. Estimation

The ML estimators enjoy desirable properties that can be used when constructing confidence intervals for the model parameters. Let $X_1, \ldots, X_n$ be a random sample of size n from $X \sim \mathrm{MC}(a, b, c, \lambda, \beta)$ with observations $x_1, \ldots, x_n$. The log-likelihood function for $\boldsymbol{\theta} = (a, b, c, \lambda, \beta)^\top$ from this sample is formulated as

$$\begin{aligned}\mathcal{L}(\boldsymbol{\theta}) =& n[\log c\lambda\beta - \log B(a,b) + \lambda] + (\beta-1)\sum_{i=1}^{n}\log x_i + \sum_{i=1}^{n}x_i^{\beta} - \lambda\sum_{i=1}^{n}\mathrm{e}^{x_i^{\beta}} \\ &+ (ac-1)\sum_{i=1}^{n}\log t(x_i) + (b-1)\sum_{i=1}^{n}\log\{1-t(x_i)^c\},\end{aligned} \tag{13}$$

where $t(x_i) = 1 - \exp\{\lambda(1-\mathrm{e}^{x_i^{\beta}})\}$.

The function $\mathcal{L}(\boldsymbol{\theta})$ can be maximized either directly by using well-known platforms such as the R (`optim` function), SAS (`PROC NLMIXED`), Ox program (`MaxBFGS` sub-routine) or by solving the nonlinear likelihood equations of the score vector obtained by differentiating Equation (13).

The components of the score vector $U(\boldsymbol{\theta})$ are given by

$$U_a(\boldsymbol{\theta}) = n\psi(a+b) - n\psi(a) + c\sum_{i=1}^{n}\log t(x_i),$$

$$U_b(\boldsymbol{\theta}) = n\psi(a+b) - n\psi(b) + \sum_{i=1}^{n}\log\{1-t(x_i)^c\},$$

$$U_c(\boldsymbol{\theta}) = \frac{n}{c} + a\sum_{i=1}^{n}\log t(x_i) - (b-1)\sum_{i=1}^{n}\frac{t(x_i)^c\log t(x_i)}{1-t(x_i)^c},$$

$$U_\lambda(\boldsymbol{\theta}) = \frac{n}{\lambda} + n - \sum_{i=1}^{n}\mathrm{e}^{x_i^{\beta}} - (ac-1)\sum_{i=1}^{n}\frac{r(x_i)}{t(x_i)} + c(b-1)\sum_{i=1}^{n}\frac{r(x_i)t(x_i)^{c-1}}{1-t(x_i)^c},$$

$$U_\beta(\boldsymbol{\theta}) = \frac{n}{\beta} + \sum_{i=1}^{n} \log x_i + \sum_{i=1}^{n} x_i^\beta \log x_i - \lambda \sum_{i=1}^{n} x_i^\beta \mathrm{e}^{x_i^\beta} \log x_i$$
$$+\lambda(ac-1)\sum_{i=1}^{n} \frac{s(x_i)}{t(x_i)} - c\lambda(b-1)\sum_{i=1}^{n} \frac{s(x_i)t(x_i)^{c-1}}{1-t(x_i)^c},$$

where $\psi(q) = \mathrm{d}\log\Gamma(q)/\mathrm{d}q$ is the digamma function, $r(x_i) = (1-\mathrm{e}^{x_i^\beta})\mathrm{e}^{\lambda(1-\mathrm{e}^{x_i^\beta})}$ and $s(x_i) = x_i^\beta\, \mathrm{e}^{\lambda(1-\mathrm{e}^{x_i^\beta})+x_i^\beta} \log x_i$.

The ML estimate $\widehat{\boldsymbol{\theta}} = (\widehat{a}, \widehat{b}, \widehat{c}, \widehat{\lambda}, \widehat{\beta})^\top$ of $\boldsymbol{\theta} = (a, b, c, \lambda, \beta)^\top$ is determined by the simultaneous solutions of the equations $U(\boldsymbol{\theta}) = \mathbf{0}$. These solutions are those $\widehat{\boldsymbol{\theta}}$ values that maximize Equation (13). The estimates of the unknown parameters can not be obtained analytically, and then interactive methods such as the quasi-Newton BFGS and Newton-Raphson algorithms are required.

The estimated observed information matrix is given by

$$J(\boldsymbol{\theta}) = -\begin{bmatrix} U_{aa}(\boldsymbol{\theta}) & U_{ab}(\boldsymbol{\theta}) & U_{ac}(\boldsymbol{\theta}) & U_{a\lambda}(\boldsymbol{\theta}) & U_{a\beta}(\boldsymbol{\theta}) \\ U_{ba}(\boldsymbol{\theta}) & U_{bb}(\boldsymbol{\theta}) & U_{bc}(\boldsymbol{\theta}) & U_{b\lambda}(\boldsymbol{\theta}) & U_{b\beta}(\boldsymbol{\theta}) \\ U_{ca}(\boldsymbol{\theta}) & U_{cb}(\boldsymbol{\theta}) & U_{cc}(\boldsymbol{\theta}) & U_{c\lambda}(\boldsymbol{\theta}) & U_{c\beta}(\boldsymbol{\theta}) \\ U_{\lambda a}(\boldsymbol{\theta}) & U_{\lambda b}(\boldsymbol{\theta}) & U_{\lambda c}(\boldsymbol{\theta}) & U_{\lambda\lambda}(\boldsymbol{\theta}) & U_{a\beta}(\boldsymbol{\theta}) \\ U_{\beta a}(\boldsymbol{\theta}) & U_{\beta b}(\boldsymbol{\theta}) & U_{\beta c}(\boldsymbol{\theta}) & U_{\beta\lambda}(\boldsymbol{\theta}) & U_{\beta\beta}(\boldsymbol{\theta}) \end{bmatrix}_{\boldsymbol{\theta}=\widehat{\boldsymbol{\theta}}},$$

where $U_{pq}(\boldsymbol{\theta}) = \partial^2\mathcal{L}(\boldsymbol{\theta})/(\partial\phi_p\partial\phi_q)$, and $U_{pq}(\boldsymbol{\theta}) = U_{qp}(\boldsymbol{\theta})$. Thus, we get

$$U_{aa}(\boldsymbol{\theta}) = n\psi'(a+b) - n\psi'(a),\ U_{ab}(\boldsymbol{\theta}) = n\psi'(a+b),\ U_{ac}(\boldsymbol{\theta}) = \sum_{i=1}^{n} \log t(x_i),$$

$$U_{a\lambda}(\boldsymbol{\theta}) = -c\sum_{i=1}^{n} \frac{r(x_i)}{t(x_i)},\ U_{a\beta}(\boldsymbol{\theta}) = c\lambda\sum_{i=1}^{n} \frac{s(x_i)}{t(x_i)},\ U_{bb}(\boldsymbol{\theta}) = n\psi'(a+b) - n\psi'(b),$$

$$U_{bc}(\boldsymbol{\theta}) = -\sum_{i=1}^{n} \frac{t(x_i)^c \log t(x_i)}{1-t(x_i)^c},\ U_{b\lambda}(\boldsymbol{\theta}) = c\sum_{i=1}^{n} \frac{r(x_i)t(x_i)^{c-1}}{1-t(x_i)^c},$$

$$U_{b\beta}(\boldsymbol{\theta}) = -c\lambda\sum_{i=1}^{n} \frac{s(x_i)t(x_i)^{c-1}}{1-t(x_i)^c},$$

$$U_{cc}(\boldsymbol{\theta}) = -\frac{n}{c^2} - (b-1)\sum_{i=1}^{n} \frac{t(x_i)^c[\log t(x_i)]^2}{[1-t(x_i)^c]^2},$$

$$U_{c\lambda}(\boldsymbol{\theta}) = -a\sum_{i=1}^{n} \frac{r(x_i)}{t(x_i)} + (b-1)\sum_{i=1}^{n} \frac{r(x_i)t(x_i)^{c-1}}{1-t(x_i)^c} + c(b-1)\sum_{i=1}^{n} \frac{r(x_i)t(x_i)^{c-1}\log t(x_i)}{[1-t(x_i)^c]^2},$$

$$U_{c\beta}(\boldsymbol{\theta}) = a\lambda\sum_{i=1}^{n} \frac{s(x_i)}{t(x_i)} - \lambda(b-1)\sum_{i=1}^{n} \frac{s(x_i)t(x_i)^{c-1}}{1-t(x_i)^c} - c\lambda(b-1)\sum_{i=1}^{n} \frac{s(x_i)t(x_i)^{c-1}\log t(x_i)}{[1-t(x_i)^c]^2},$$

$$U_{\lambda\lambda}(\boldsymbol{\theta}) = -\frac{n}{\lambda^2} - (ac-1)\sum_{i=1}^{n} \frac{(1-\mathrm{e}^{x_i^\beta})r(x_i)t(x_i) + r(x_i)^2}{t(x_i)^2} - c^2(b-1)\sum_{i=1}^{n} \frac{[r(x_i)t(x_i)^{c-1}]^2}{[1-t(x_i)^c]^2}$$
$$+c(b-1)\sum_{i=1}^{n} \frac{(1-\mathrm{e}^{x_i^\beta})r(x_i)t(x_i)^{c-1} - (c-1)r(x_i)^2 t(x_i)^{c-2}}{1-t(x_i)^c},$$

$$
\begin{aligned}
U_{\lambda\beta}(\boldsymbol{\theta}) = & -\sum_{i=1}^{n} x_i^{\beta} \mathrm{e}^{x_i^{\beta}} \log x_i + (ac-1)\sum_{i=1}^{n} \frac{s(x_i)}{t(x_i)} \\
& + c^2\lambda(b-1)\sum_{i=1}^{n} \frac{s(x_i)r(x_i)t(x_i)^{2c-2}}{[1-t(x_i)^c]^2} \\
& + \lambda(ac-1)\sum_{i=1}^{n} \frac{(1-\mathrm{e}^{x_i^{\beta}})s(x_i)t(x_i) + s(x_i)r(x_i)}{t(x_i)^2} \\
& - c(b-1)\sum_{i=1}^{n} \frac{s(x_i)t(x_i)^{c-1}}{1-t(x_i)^c} \\
& - c\lambda(b-1)\sum_{i=1}^{n} \frac{(1-\mathrm{e}^{x_i^{\beta}})s(x_i)t(x_i)^{c-1} - (c-1)s(x_i)r(x_i)t(x_i)^{c-2}}{1-t(x_i)^c}, \\
U_{\beta\beta}(\boldsymbol{\theta}) = & -\frac{n}{\beta^2} + \sum_{i=1}^{n} x_i^{\beta}(\log x_i)^2 - \lambda\sum_{i=1}^{n} x_i^{\beta}\mathrm{e}^{x_i^{\beta}}(\log x_i)^2[1+x_i^{\beta}] \\
& + \lambda(ac-1)\sum_{i=1}^{n} \frac{v(x_i)t(x_i) - \lambda s(x_i)^2}{t(x_i)^2} \\
& - c\lambda(b-1)\sum_{i=1}^{n} \frac{v(x_i)t(x_i)^{c-1} + \lambda(c-1)s(x_i)^2 t(x_i)^{c-2}}{1-t(x_i)^c} \\
& - (c\lambda)^2(b-1)\sum_{i=1}^{n} \frac{[s(x_i)t(x_i)^{c-1}]^2}{[1-t(x_i)^c]^2},
\end{aligned}
$$

where $\psi'(q) = \mathrm{d}^2 \log\Gamma(q)/\mathrm{d}q^2$ is the trigamma function and $v(x_i) = s(x_i)[\log x_i - \lambda x_i^{\beta}\mathrm{e}^{x_i^{\beta}} \log x_i + x_i^{\beta} \log x_i]$

The normal approximation for $\widehat{\boldsymbol{\theta}}$ in distribution theory is easily handled numerically. Under general regularity conditions, we have the result $(\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}) \overset{a}{\sim} \mathrm{N}_5(\mathbf{0}, \mathbf{K}(\boldsymbol{\theta})^{-\mathbf{1}})$, where $K(\boldsymbol{\theta})$ is the 5×5 expected information matrix and $\overset{a}{\sim}$ denotes asymptotic distribution. For n large, $K(\boldsymbol{\theta})$ can be approximated by the estimated observed information matrix $J(\widehat{\boldsymbol{\theta}})$. This multivariate normal approximation for $\widehat{\boldsymbol{\theta}}$ can be used for construing approximate confidence intervals for the model parameters. The LR statistics can be used for testing hypotheses on these parameters.

7. Simulation study

A Monte Carlo simulation is performed to empirically evaluate some asymptotic properties of the ML estimators for the parameters of the MC distribution. The MC observations are generated from three different combinations of a, b, c, λ and β with samples sizes $n = 25, 50, 75, 100, 200$ and 500 and repeat the simulations $N = 1,000$ times. The subroutine `optim` in `R` (R Core Team, 2020) is used for maximizing the log-likelihood Equation (13). The average estimates (AEs) of the ML estimators and their mean squared errors (MSEs) are reported in Tables 1, 2 and 3. The AEs tend to be closer to the true parameters and the MSEs decrease when the sample size n increases in agreement with first-order asymptotic theory. Note that the parameter β presents the lowest MSE in all scenarios. In addition, the parameter λ is the one which presents the highest MSE.

Table 1. Monte Carlo results under $\boldsymbol{\theta} = (1.3, 1.6, 1.4, 1.2, 0.6)$.

	AE					MSE				
n	$\widehat{a}$	$\widehat{b}$	$\widehat{c}$	$\widehat{\lambda}$	$\widehat{\beta}$	$\widehat{a}$	$\widehat{b}$	$\widehat{c}$	$\widehat{\lambda}$	$\widehat{\beta}$
25	2.219	2.102	2.916	2.958	1.483	7.706	6.801	10.945	11.097	3.319
50	1.942	1.913	2.658	2.689	0.946	5.256	4.602	7.705	8.133	0.728
75	1.834	1.927	2.563	2.423	0.835	4.197	4.062	6.320	5.910	0.348
100	1.908	1.788	2.403	2.399	0.788	4.251	3.314	5.917	5.272	0.218
200	1.726	1.805	2.133	2.089	0.681	2.557	2.654	3.500	3.389	0.053
500	1.630	1.770	1.892	1.717	0.632	1.727	1.712	2.160	1.511	0.013

Table 2. Monte Carlo results under $\boldsymbol{\theta} = (1.4, 2, 0.9, 2.8, 1.1)$.

	AE					MSE				
n	$\widehat{a}$	$\widehat{b}$	$\widehat{c}$	$\widehat{\lambda}$	$\widehat{\beta}$	$\widehat{a}$	$\widehat{b}$	$\widehat{c}$	$\widehat{\lambda}$	$\widehat{\beta}$
25	2.173	3.919	2.110	5.153	2.497	6.493	12.609	8.104	19.458	6.186
50	2.028	3.514	1.822	4.353	2.251	4.719	9.481	6.205	13.103	4.631
75	1.932	3.381	1.612	3.958	2.180	3.297	7.829	4.690	10.588	4.319
100	1.911	3.271	1.614	3.701	1.983	3.407	6.675	4.481	8.492	3.269
200	1.904	3.025	1.443	3.328	1.792	2.845	4.440	2.989	6.200	2.297
500	1.818	2.835	1.347	2.916	1.405	1.972	3.225	2.057	3.555	0.808

Table 3. Monte Carlo results under $\boldsymbol{\theta} = (1.7, 1.9, 1.2, 2.2, 0.7)$.

	AE					MSE				
n	$\widehat{a}$	$\widehat{b}$	$\widehat{c}$	$\widehat{\lambda}$	$\widehat{\beta}$	$\widehat{a}$	$\widehat{b}$	$\widehat{c}$	$\widehat{\lambda}$	$\widehat{\beta}$
25	2.689	3.342	2.302	4.495	2.056	8.880	10.736	8.074	17.959	5.889
50	2.367	3.046	2.132	3.610	1.700	5.293	7.422	6.402	11.657	3.954
75	2.262	2.859	2.094	3.479	1.471	4.589	6.739	5.869	10.489	2.886
100	2.297	2.734	1.897	3.279	1.263	4.167	5.378	4.320	8.920	1.844
200	2.209	2.599	1.925	2.913	0.968	3.824	4.000	3.793	5.655	0.638
500	2.087	2.412	1.694	2.609	0.798	2.235	2.489	2.114	2.963	0.187

8. Applications

Two real data applications prove empirically the adequacy of the MC distribution. The applications are developed using the `R` software (version 3.6.3) (R Core Team, 2020) with the script `AdequacyModel` (Marinho et al., 2019). The criteria for model selection are based on the statistics defined by Chen and Balakrishnan (1995): Anderson Darling (A^*) and Cramér-von Mises (W^*). In addition to these statistics, we consider the Akaike information criterion (AIC), Consistent Akaike information criterion (CAIC), Bayesian information criterion (BIC), Hannan-Quinn information criterion (HQIC) and Kolmogorov-Smirnov (KS) statistic with its p-value for model comparisons. The smaller the value of these statistics evidence we have for a good fit. All these important statistics for selecting the best models are provided in the `AdequacyModel` package. The graphical analysis is also important to identify the best fitted model. We analyze the data histograms, the estimated PDFs and CDFs and the empirical CDF calculated by the Kaplan-Meier (Kaplan and Meier, 1958) method.

The MC distribution is compared with three popular lifetime models. The first one is the beta-modified Weibull (BMW) distribution defined by Silva et al. (2010), whose PDF is given by

$$f_{\mathrm{BMW}}(x;a,b,\alpha,\lambda,\gamma)=\frac{a x^{\gamma-1}(\gamma+\lambda x)\mathrm{e}^{\lambda x}}{B(a,b)}\mathrm{e}^{-b\alpha x^{\gamma}\mathrm{e}^{\lambda x}}\left[1-\mathrm{e}^{-\alpha x^{\gamma}\mathrm{e}^{\lambda x}}\right]^{a-1},\quad x>0,$$

where a, b, and γ are positive shape parameters, $\alpha > 0$ is a scale parameter and $\lambda > 0$ is an accelerating factor in imperfection time which acts as a fragility factor in the survival of the individual as time increases.

The second one is the three-parameter Burr XII distribution (Zimmer et al., 1998), whose PDF has the form

$$f_{\mathrm{BXII}}(x;s,d,c)=\frac{cd}{s^{c}}x^{c-1}\left[1+\left(\frac{x}{s}\right)^{c}\right]^{-(d+1)},\quad x>0,$$

where $s > 0$ is a scale parameter and c and d are two positive shape parameters.

The third distribution is the Kumaraswamy-log logistic (KLL) (de Santana et al., 2012) model, whose PDF is stated as

$$f_{\mathrm{KLL}}(x;a,b,\alpha,\gamma)=\frac{ab\gamma}{\alpha^{a\gamma}}x^{a\gamma-1}\left[1+\left(\frac{x}{\alpha}\right)^{\gamma}\right]^{-(a+1)}\left\{1-\left[1-\frac{1}{1+(\frac{x}{\alpha})^{\gamma}}\right]^{\alpha}\right\}^{b-1},\quad x>0,$$

where $\alpha > 0$ is a scale parameter and a, b and γ are positive shape parameters.

8.1 Windshields data

We consider 85 uncensored failure times for a specific windshield model studied by Murthy et al. (2004) and Cordeiro et al. (2015). A problem of interest would be to accurately estimate the probability of failure of this windshield model within a specified period time.

The descriptive statistics for these data are listed in Table 4, including minimum and maximum values, first and third quartile, median (Med), mean, standard deviation (SD), and coefficients of skewness and kurtosis.

Table 4. Descriptive statistics for windshields data.

n	Min	1st quartile	Med	Mean	3rd quartile	Max	SD	Skewness	Kurtosis
85	0.04	1.87	2.38	2.56	3.38	4.66	1.11	0.09	2.37

The ML estimates and their associated standard errors (SEs) in parentheses for the fitted distributions are reported in Table 5. Some estimators have large SEs for the BMW and BXII distributions. In addition, the MC and KLL distribution parameters are significant. Table 6 gives the values of the information criteria described before. The MC distribution has the lowest values for all information criteria. Thus, it is the distribution that yields the best fit to the current data. The p-values of the KS statistic also reveal that the data are described well for all distributions.

Since the MG family includes as special cases the beta-G and Kumaraswamy-G classes, two LR tests are performed: MC versus beta-Chen ($c = 1$) and MC vs Kumasrawamy-Chen ($a = 1$). The LR statistics for these tests are 7.0369 and 7.2441, respectively. The two null hypotheses are rejected, thus indicating that the MC distribution is the most suitable for the current data.

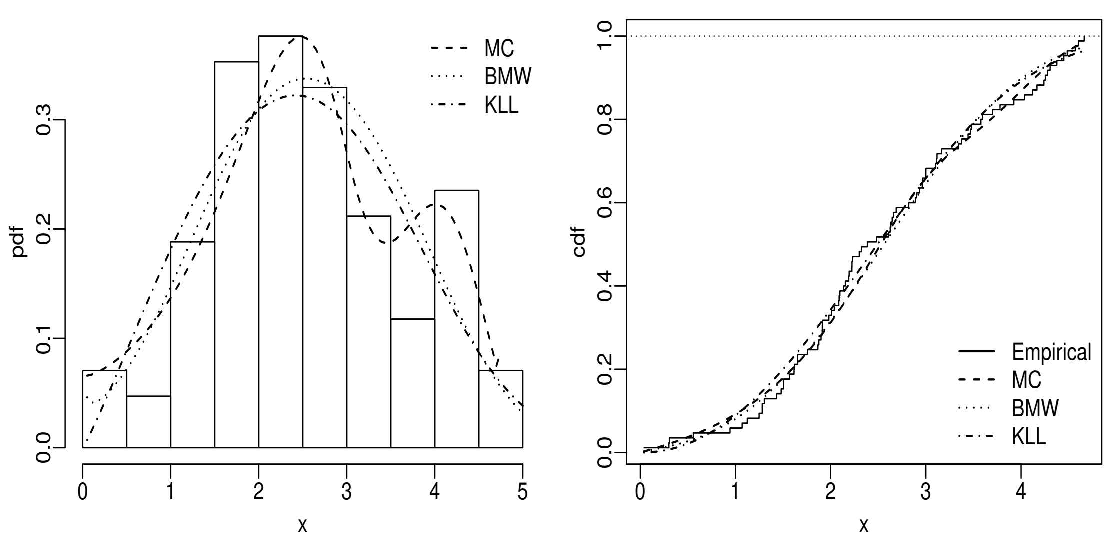
Table 5. ML estimates and their associated standard errors (SEs) in parentheses for the distributions fitted to windshields data.

Distribution	Estimate				
MC$(a, b, c, \lambda, \beta)$	0.0360	0.0994	22.2596	0.0451	1.2201
	(0.0051)	(0.0228)	(0.6653)	(0.0186)	(0.0139)
BMW$(a, b, \alpha, \lambda, \gamma)$	4.9756	0.1824	1.0938	0.6004	0.1290
	(5.2775)	(0.1640)	(0.7173)	(0.1519)	(0.1963)
KLL(a, b, α, γ)	0.3359	3.5033	5.6294	6.2686	
	(0.0374)	(0.6590)	(0.0304)	(0.0306)	
BXII(s, d, c)	13.5330	42.7872	2.4122		
	(8.5003)	(59.8242)	(0.2171)		

Table 6. Statistics for the fitted distributions to windshields data.

Distribution	W^*	A^*	AIC	CAIC	BIC	HQIC	KS	p-value (KS)
MC	0.0576	0.3684	259.0505	259.8100	271.2638	263.9630	0.0810	0.6332
BMW	0.0683	0.4849	264.4261	265.1856	276.6394	269.3387	0.0820	0.6173
KLL	0.0617	0.5597	268.1631	268.6631	277.9337	272.0931	0.0570	0.9454
BXII	0.0590	0.5973	269.0118	269.3081	276.3398	271.9594	0.0538	0.9663

A graphical analysis can show the best choice for a model. First, the estimated PDFs are plotted on the data histogram in Figure 5(a). These plots show that the MC distribution is the most appropriate model for the current data and that its estimated PDFs captures the bimodality of the histogram. Figure 5(b) displays the empirical CDF and the estimated CDFs of the MC, BMW and KLL models, which also reveals the superiority of the MC distribution for these data.



(a) (b)

Figure 5. Histogram with estimated PDFs (a) and empirical CDF with estimated CDFs (b) of the MC, BMW and KLL models for windshields data.

8.2 Kevlar/epoxy data

This data set is about the lifetime of spherical pressure vessels under constant pressure until vessel failure, commonly known as static fatigue or stress rupture. NASA space shuttle uses Kevlar/epoxy spherical pressure vessels in a sustained pressure mode for the life of the vessel. The use of this material can be found in air-space breathing apparatus. These data are available in Andrews and Herzberg (1985). The main interest in this application would be to accurately estimate the survival function of these spherical pressure vessels.

The descriptive statistics for these data are given in Table 7. The ML estimates of the parameters for four fitted models are listed in Table 8. Again, the BMW and BXII distributions have large SEs for some estimates. Differently, all the MC and KLL parameters are significant.

Table 7. Descriptive statistics for Kevlar/epoxy data.

n	Min	1st quartile	Med	Mean	3rd quartile	Max	SD	Skewness	Kurtosis
49	1051	5620	8831	8805.69	11745	17568	4553.92	0.10	2.17

Table 8. ML estimates and their associated standard errors (SEs) in parentheses for the distributions fitted to Kevlar/epóxy data.

Distribution	Estimate				
MC$(a, b, c, \lambda, \beta)$	0.3290	0.1171	5.3114	0.1595	0.5822
	(0.0758)	(0.0248)	(0.0380)	(0.0137)	(0.0114)
BMW$(a, b, \alpha, \lambda, \gamma)$	0.7204	0.4003	0.0162	0.0814	1.7067
	(0.5976)	(1.1377)	(0.0308)	(0.1062)	(1.6739)
KLL(a, b, α, γ)	0.2718	13.0771	37.9120	7.1778	
	(0.0433)	(4.5558)	(0.4090)	(0.6379)	
BXII(s, d, c)	39.5646	18.5077	2.0830		
	(28.7962)	(25.6918)	(0.2439)		

The LR values for the tests MC vs BC ($c = 1$) and MC vs KC ($a = 1$) are 1.1182 and 0.8931, respectively, and therefore the two null hypotheses are not rejected. In Table 9, the more useful statistics W^* and A^* to compare nested and non-nested models indicate that the MC distribution is more appropriate for the current data. The KC distribution can also be chosen based on the AIC, CAIC and HQIC criteria. According to BIC criteria the BXII model is chosen. However, these criterions are more useful to compare nested models. The p-values of the KS statistic indicate that all models can be adopted to fit the current data, although it is higher for the BXII model.

Table 9. Statistics for the fitted models to Kevlar/epóxy data.

Distribution	W^*	A^*	AIC	CAIC	BIC	HQIC	KS	p-value (KS)
MC	0.0294	0.2228	291.3719	292.7673	300.8310	294.9607	0.0724	0.9593
BMW	0.0313	0.2304	291.6484	293.0438	301.1075	295.2372	0.0697	0.9711
KLL	0.0639	0.4196	291.8350	292.7441	299.4023	294.7060	0.0849	0.8716
BXII	0.0800	0.5221	291.4196	291.9530	297.0951	293.5729	0.0902	0.8198
BC	0.0350	0.2491	290.4901	291.3992	298.0574	293.3612	0.0750	0.9455
KC	0.0354	0.2494	290.2651	291.1742	297.8324	293.1361	0.0749	0.9463

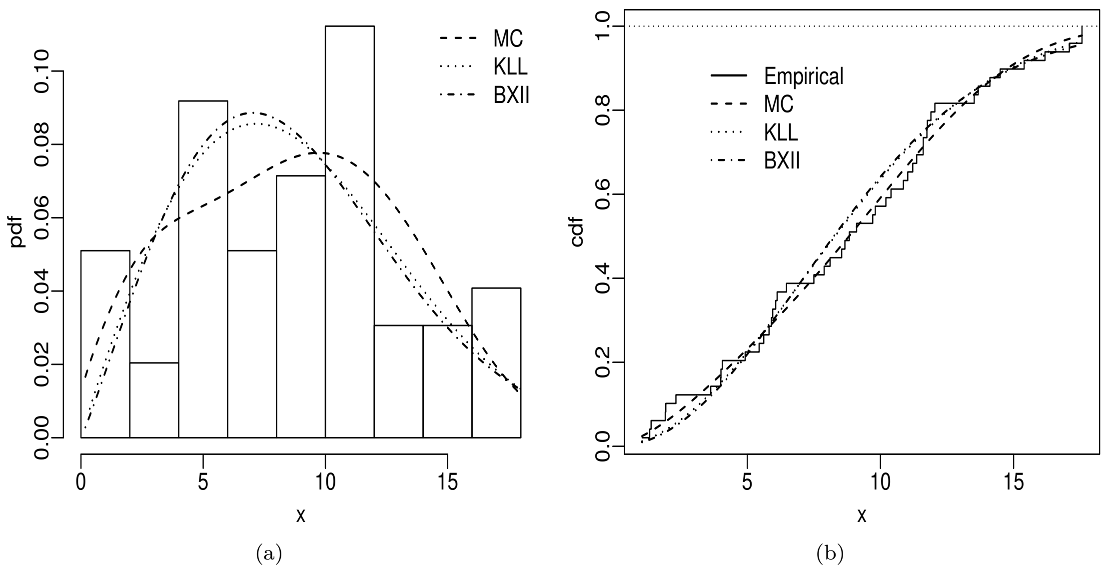


Figure 6. Histogram with estimated PDFs (a) and empirical CDF with estimated CDFs (b) of the MC, KLL and BXII models for Kevlar/epoxy data.

The histogram and estimated PDFs are reported in Figure 6(a), where the superiority of the MC distribution is noted, thus corroborating with the W^* and A^* statistics.

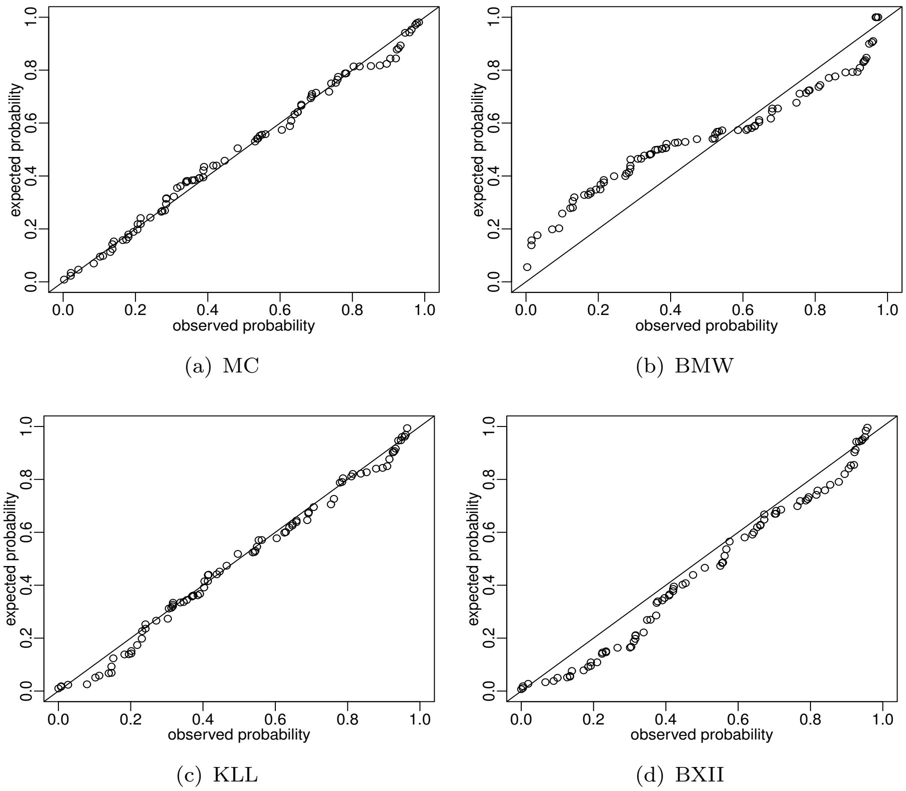
The estimated CDFs along and the empirical CDF are displayed in Figure 6(b). These plots reveal that the estimated CDF of the MC model is closer to the empirical one. Thus, the MC model has a better performance to explain the survival function of the data.

The probability-probability (PP) plots for windshields and Kevlar/epóxy data are given in Figures 7 and 8, respectively. For both data sets, the plot points are close to the diagonal line for the MC model, followed by the KLL distribution. This is a further indication that the MC distribution is the best model for these data sets. Plot of the profile log-likelihood function for windshields and Kevlar/epóxy data are shown in Figures 9 and 10, respectively. These plots were constructed by fixing the other parameters and varying the parameter of interest in a range covering the respective ML estimate. For example in Figure 9(a), the parameter a varies between 0.01 and 0.2, in Figure 9(b) $0.01 < b < 0.8$, in Figure 9(c) $10 < c < 40$, in Figure 9(d) $0.004 < \lambda < 0.05$ and Figure 9(e) $0.3 < \beta < 1.24$. The plots of the Figure 10 are constructed in an analogous way.

9. Conclusions, limitations, and future research

In this paper, the new McDonald-Chen distribution was proposed, which extended the Chen distribution and presented more flexibility. In the proposal, three shape parameters were added to the Chen distribution to obtain more flexibility and bimodality for the generated probability density function. Its failure or hazard rate function can be increasing, decreasing, upside-down bathtub, bathtub and increasing-decreasing-increasing shapes. Few distributions have this last form. As a result, the new distribution can accommodate several types of data sets, so providing a good alternative for fitting survival and fatigue data. Monte Carlo simulations evaluated the accuracy of the maximum likelihood estimators of the parameters. Finally, two real applications showed that the McDonald-Chen distribution provided better fits than three well-known models because it accommodates bimodality.

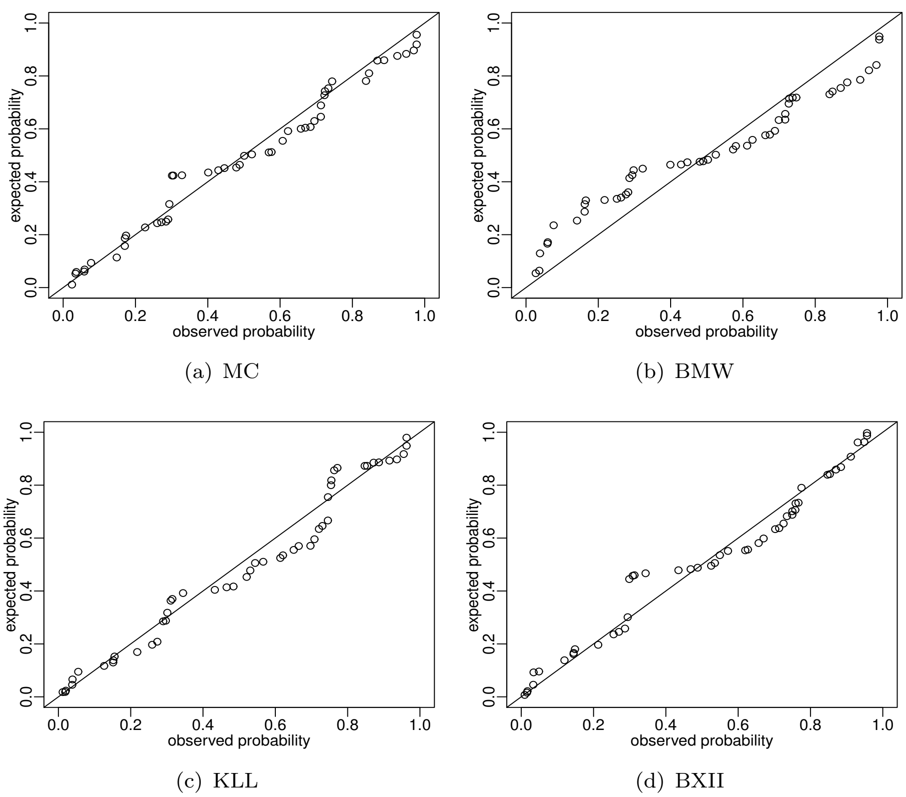
A limitation of the new distribution proposed here is its usefulness in fitting data with very small samples because this distribution has five parameters. This compromises the degrees of freedom for data with small samples.



(a) MC (b) BMW

(c) KLL (d) BXII

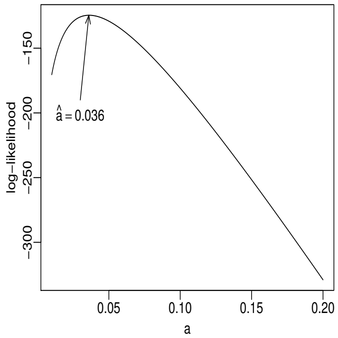
Figure 7. PP-plots for windshields data.



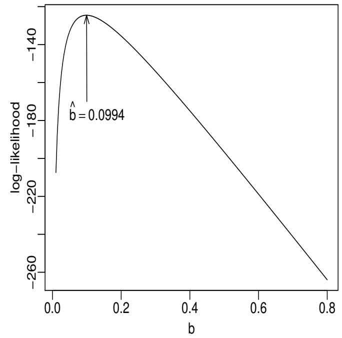
(a) MC (b) BMW

(c) KLL (d) BXII

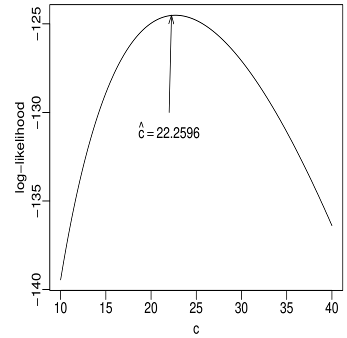
Figure 8. PP-plots for Kevlar/epóxy data.



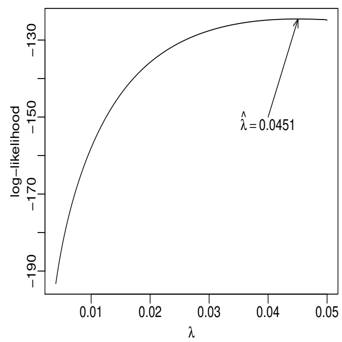
(a) ML estimate of a



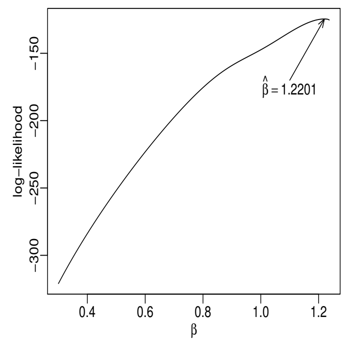
(b) ML estimate of b



(c) ML estimate of c

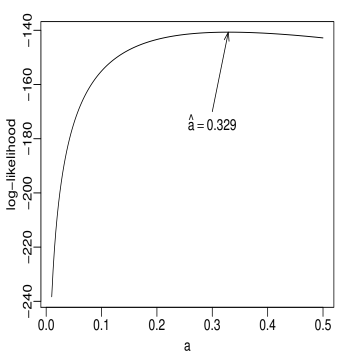


(d) ML estimate of λ

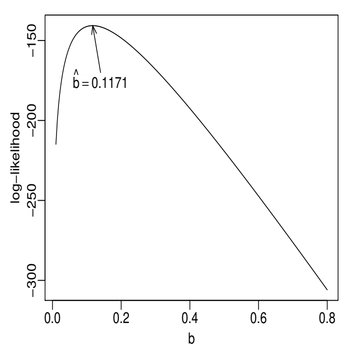


(e) ML estimate of β

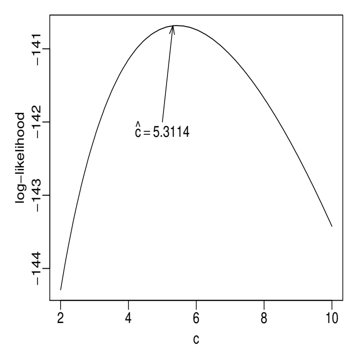
Figure 9. Profile log-likelihood functions for windshields data.



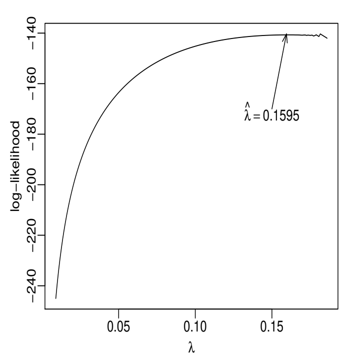
(a) ML estimate of a



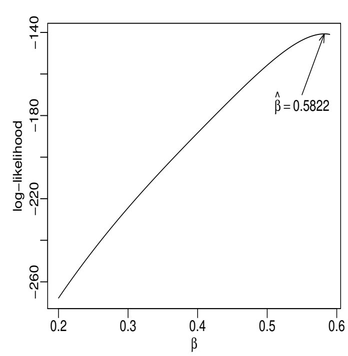
(b) ML estimate of b



(c) ML estimate of c



(d) ML estimate of λ



(e) ML estimate of β

Figure 10. Profile log-likelihood functions for Kevlar/epóxy data.

Future work can be directed to: (i) correct the maximum likelihood estimators analytically (if possible) or numerically (via bootstrap resampling); (ii) reparameterize the McDonald-Chen distribution in terms of the median and propose a regression model to model the median; and (iii) perform inference studies on the McDonald-Chen regression model and diagnostic analysis.

Author contributions Conceptualization, L.D.R.R., G.M.C., J.J.S.S.; methodology, L.D.R.R., G.M.C., J.J.S.S.; software, L.D.R.R., J.J.S.S.; validation, L.D.R.R., G.M.C., J.J.S.S.; formal analysis, L.D.R.R., J.J.S.S.; investigation, L.D.R.R., J.J.S.S.; data curation, L.D.R.R., J.J.S.S.; writing-original draft preparation, L.D.R.R., J.J.S.S.; writing-review and editing, G.M.C.; visualization, L.D.R.R., J.J.S.S., G.M.C.; supervision, G.M.C. All authors have read and agreed the published version of the paper.

Acknowledgements The authors would also like to thank the Editors-in-Chief and the anonymous reviewers for comments that improved the paper.

Funding The authors would like to thank the Fundação de Amparo à Ciência e Tecnologia do Estado de Pernambuco (FACEPE), Brazil.

Conflicts of interest The authors declare no conflict of interest.

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