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Aims

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Time Series
Research Paper

A Bayesian detection of structural changes in autoregressive time series models

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Abstract

This study investigates a Bayesian detection of a change in any parameter, or in any collection of parameters of the autoregressive time series model of known order p. An unconditional Bayesian test based on the highest posterior density credible sets is determined. Using the Gibbs sampler algorithm, some simulated results are given to approximate the posterior densities of the change point and other parameters of the model. The performance of our proposed method has been investigated on simulated and real data sets.

Keywords: Bayesian analysis· Change point· Gibbs sampler· HPD credible set· p-value.

Mathematics Subject Classification: Primary 62M10 · Secondary 62F15.

1. Introduction

Change point detection is an important element in time series analysis that arises in many fields such as quality control procedures (Basseville and Nikiforov (1993)), anomaly detection in internet traffic data (Lévy-Leduc and Roueff, 2009; Tartakovsky et al., 2006), metrology (Jandhyala et al., 2014), economics and financial analysis (Georgescu, 2012), and biology (Fan et al., 2015), among others. Change point detection is the problem of detecting abrupt changes in the parameters of temporal or other sequential data. Since the papers of Page (1954) and Page (1955), who proposed a sequential scheme for identifying changes in the mean of a sequence of independent random variables, the problem of detecting changes has been an important issue between statisticians and considerable attention has been given to this problem in a variety of settings. For example, changes in a sequence of random variables have been considered by Eastwood (1993), Gombay and Horvath (1999) and Guo and Modarres (2020) from the nonparametric viewpoint. Montoril and da Silva Ferreira (2018) proposed a method based on the coefficient of determination, to estimate the change points in the Beer-Lambert law problems. Among the approach based on likelihood ratio, Worsley (1983, 1986) proposed a numerical method for computing the p-value of the generalized likelihood ratio test to detect a change in the binomial probability and in the location of an

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exponential family distribution. Kim (1996) considered a likelihood ratio test for a change in the mean when observations are correlated. Kim and Siegmund (1989) considered likelihood ratio tests to detect a change-point in simple linear regression. Wang et al. (2020) used the likelihood ratio test to detect changes in the parameters of the skew slash distribution.

From a Bayesian point of view, the problem of detecting a change has received much attention and has been studied by many authors like Chernoff and Zacks (1964), Kander and Zacks (1966), Sen and Srivastava (1975), Jani and Pandya (1999), Fan and Chen (2005) and Shah and Patel (2007). Ming Ng (1990) analyzed a linear model in which both the mean and the precision change once at an unknown time point, the posterior distributions of the change point, and the ratio of the precisions are derived.

Kim (1991) proposed a Bayesian significance test for the stationarity of a regression equation using the highest posterior density (HPD) credible set. From a Monte Carlo simulation study, he showed that the Bayesian significance test has a stronger power than the Cusum and the Cusum of squares tests suggested by Brown et al. (1975). Sáfadi and Morettin (2000) considered a Bayesian analysis for threshold autoregressive moving average models. Pan et al. (2017) considered a Bayesian analysis of threshold autoregressive (TAR) model with various possible thresholds. Recently, Hahn et al. (2020) introduced a computationally inexpensive Bayesian approach (BayesProject) for detecting changes in mean within multivariate data sequences.

For autoregressive time series models, many papers about detecting and estimating changes in autoregressive time series of known order p (AR(p)) processes have been published. For example, Davis et al. (1995) studied the asymptotic behavior of a Gaussian-type likelihood ratio statistic for testing a change in the parameters of an AR(p) model. Husková et al. (2007, 2008) used an approach based on partial sums of weighted residuals (asymptotic and bootstrapping methods). Venkatesan and Arumugam (2007) considered the problem of gradual changes in the parameters of an autoregressive time series model. Gombay (2008) used the efficient score vector to detect change in the parameter(s) of autoregressive time series. Berkes et al. (2011) developed the likelihood ratio test for the structural change of an AR model to a threshold AR model. Slama (2014) examined the effect of correlation on the performance of the Bayesian significance test derived under the assumption of no correlation. By numerical studies, he showed that the Bayesian significance test based on the HPD region is sensitive to the correlation in the data. Kezim and Abdelli (2004) proposed a Bayesian analysis of a first order autoregressive process subject to one change in both the variance of the error terms and the autocorrelation coefficients at an unknown time point. The detection of possible changes in the parameters of autoregressive models for binary time series can be found in Hudecová (2013). Cheon and Kim (2014) proposed a general solution to detect the Bayesian estimation in Bayesian autoregressive structural-change time series models. A Bayesian approach to estimate the multiple structural change-points in a level and the trend when the number of change-points is unknown was proposed. Slama and Saggou (2017) investigated the Bayesian approach using HPD credibles sets and p-values for detecting an abrupt change in the parameters of an AR(p). In a recent work, Gamage and Ning (2021) proposed a nonparametric method based on the empirical likelihood is proposed to detect the structural changes in the autoregressive parameters of autoregressive models. In the last three works, the mean is assumed constant and equal to 0.

Bauwens et al. (2014) solved the problem of the computation of the marginal likelihood for a Markov-switching GARCH or change-point GARCH models by applying a particle Markov chain Monte Carlo (PMCMC) method. Recently, Romano et al. (2021) proposed a principled approach to detect abrupt changes in mean in univariate time-series that models local fluctuations as a random walk process and autocorrelated noise via an AR(1) process. For a review of methods of inference for single and multiple change-points in time series, we refer the reader to Jandhyala et al. (2013) and Truong et al. (2020).

In this paper, we investigate a Bayesian detection of a change in any parameter, or in any collection of parameters of an AR(p). We consider a Bayesian significance test for an abrupt change at an unknown time point in the mean, the autocorrelation coefficients and the variance of the error terms of an AR(p). This work is an extension of the paper by Slama and Saggou (2017) to the case where the mean is unknown and changes at an unknown time.

The rest of the paper is organized as follows. Section 2 presents the model AR(p) with change in the parameters at an unknown time point and some notations used along this paper. In Section 3 we give the conditional posterior distributions of the parameters of change and Bayesian significance test of change in AR(p) model. In Section 4 we present a simulation results with the application of the Gibbs sampler algorithm. A real data analysis is provided in Section 5. Finally, our conclusion is presented in Section 6.

2. Definition of the Model and notations

Assume that we observe a real time series, $y_1, \ldots, y_n$ namely, generated from an AR(p) model, with a change in the mean μ, the autocorrelation coefficients ϕ_i and in the variance σ^2 at an unknown time point m. The AR(p) model with structural change is given by

$$Y_t - \mu_1 = \sum_{i=1}^{p} \phi_i \left(Y_{t-i} - \mu_1\right) + \epsilon_t, \quad t = 1, \ldots, m, \tag{1}$$

$$Y_t - \mu_2 = \sum_{i=1}^{p} \psi_i \left(Y_{t-i} - \gamma_{t-i}\mu_1 - (1-\gamma_{t-i})\mu_2\right) + \epsilon_t, \quad t = m+1, \ldots, m+p,$$

$$Y_t - \mu_2 = \sum_{i=1}^{p} \psi_i (Y_{t-1} - \mu_2) + \epsilon_t, \quad t = m+p+1, \ldots, n,$$

where γ_t is the indicator function such that $\gamma_{t-i} = 1$ if $t - i \leq m$ and $\gamma_{t-i} = 0$ if $t - i > m$. $\epsilon_t \sim \mathrm{N}(0, \sigma_1^2)$, for $t = 1, \ldots, m$ and $\epsilon_t \sim \mathrm{N}(0, \sigma_2^2)$, for $t = m+1, \ldots, n$. The parameters $\mu_i \in \mathrm{R}$, $\sigma_i > 0$), for $i = 1, 2$, and ϕ_i, ψ_i, for $i = 1, \ldots, p$, are assumed to be unknown, and $m = 1, \ldots, n-2$ is the change point assumed also unknown. If $\phi_i \neq \psi_i$ for some $i = 1, \ldots, p$, the structure of the series has changed from an AR(p) model with coefficient ϕ_i to another AR(p) model with coefficient ψ_i. We assume that the autoregressive parameters correspond to stationary processes in the sense that the parameter vector $\phi^{(p)} = (\phi_1, \phi_2, \ldots, \phi_p)$ lies in the stationary region $\Phi_1^{(p)} = \{z/1 - \phi_1 z - \phi_2 z^2 - \cdots - \phi_p z^p = 0\}$, which implies $|z| > 1$, and likewise $\psi^{(p)} = (\psi_1, \psi_2, \ldots, \psi_p)$ lies in the stationary region $\Phi_2^{(p)} = \{z/1 - \psi_1 z - \psi_2 z^2 - \cdots - \psi_p z^p = 0\}$, which implies $|z| > 1$. The quantities $y_{1-p}, \ldots, y_{-1}, y_0$ are the initial observations assumed to be stated.

The model given in Equation (1) is more general than the model considered in Slama and Saggou (2017). In Slama and Saggou (2017), the mean μ is assumed to be constant and equal to 0. Whereas, in Equation (1) the mean is assumed unknown and changes at an unknown time point m, which increases the size of the parameter space. The parameter space for the model given in Equation (1) is $\Theta = \{\theta = (m, \mu_1, \mu_2, \phi_1, \phi_2, \ldots, \phi_p, \psi_1, \ldots, \psi_p, r_1, r_2)\}$, where $r_i = 1/\sigma_i^2, i = 1, 2$, with $m = 1, \ldots, n-2, \mu_1, \mu_2 \in \mathrm{R}, r_1, r_2 \in \mathrm{R}_+^*$, and $\phi_i, \psi_i \in \Phi^{(p)}$, for $i = 1, \ldots, p$.

We want to test whether or not a change-point occurs in the autoregressive parameters. Thus, we build an inference about testing the hypotheses: H_0: $\delta = \mu_2 - \mu_1 = 0$ and $\rho_j = \psi_j - \phi_j = 0, \forall j = 1, \ldots, p$ and $\tau = \sigma_2^2/\sigma_1^2 = 1$, against H_1: $\delta = \mu_2 - \mu_1 \neq 0$ or for at least one $\rho_j = \psi_j - \phi_j \neq 0, j = 1, \ldots, p$, or $\tau = \sigma_2^2/\sigma_1^2 \neq 1$. Hence, under the alternative hypothesis, there is a change in at least one of the $2p+5$ parameters at an unknown time point. The proposed test is based on the posterior distribution of the shift $\delta = \mu_2 - \mu_1$, $\rho_j = \psi_j - \phi_j$ and of the ratio $\tau = \sigma_2^2/\sigma_1^2$. The hypothesis meaning "no change" is equivalent to H_0': $m = n$ and H_1 is equivalent to $H_1' : m \neq n$.

For the rest of the paper, we consider the notations: $\phi^{(p)} = (\phi_1, \ldots, \phi_p)$, $\psi^{(p)} = (\psi_1, \ldots, \psi_p)$, $\rho^{(p)} = (\rho_1, \ldots, \rho_p)$, $\phi^{(-j)} = (\phi_1, \ldots, \phi_{j-1}, \phi_{j+1}, \ldots, \phi_p)$ and $\rho^{(-j)} = (\rho_1, \ldots, \rho_{j-1}, \rho_{j+1}, \ldots, \rho_p)$, where $\rho_j = \psi_j - \phi_j$, $j = 1, \ldots, p$. The functional forms $\pi(\cdot)$ and $\pi(\cdot \mid \cdot)$ represent a prior and a posterior distribution, respectively.

The parameter set $\theta = (m, \mu_1, \mu_2, \phi^{(p)}, \psi^{(p)}, r_1, r_2)$, where $r_i = 1/\sigma_i^2$, for $i = 1, 2$, is a vector of dimension $(2p+5)$. The conditional likelihood function based on the observations $y = (y_1, \ldots, y_n)$ is given by

$$\begin{aligned} l(y|\theta) \propto \quad & r_1^{\frac{m}{2}} r_2^{\frac{n-m}{2}} \exp\left\{ -\frac{r_1}{2} \left[\sum_{t=1}^{m} (y_t - \mu_1 - \sum_{i=1}^{p} \phi_i (y_{t-i} - \mu_1) \right]^2 \right\} \\ & \exp\left\{ -\frac{r_2}{2} \left[\sum_{t=m+1}^{m+p} (y_t - \mu_2 - \sum_{i=1}^{p} \psi_i \Big(y_{t-i} - \gamma_{t-i}\mu_1 - (1-\gamma_{t-i})\mu_2 \Big) \right]^2 \right\} \\ & \exp\left\{ -\frac{r_2}{2} \left[\sum_{t=m+p+1}^{n} (y_t - \mu_2 - \sum_{i=1}^{p} \psi_i (y_{t-i} - \mu_2) \right]^2 \right\}. \end{aligned} \tag{2}$$

The conditional likelihood approach is based on the assumption that the initial observations $y_0, y_{-1}, \ldots, y_{1-p}$ are also available (Reinsel, 1997). Moreover, if the sample size n is sufficiently large, the first observation makes a negligible contribution to the total likelihood (Hamilton, 1994).

3. Bayesian analysis

In this section, the conditional posterior distribution of the shift in the mean δ, in the autocorrelation coefficients ρ_j, $j = 1, \ldots, p$, of the variance ratio τ and of the change point m are derived. These distributions are used to define an unconditional Bayesian significance test of change in the parameters of an AR(p).

Since prior knowledge of $\theta' = (\mu_1, \mu_2, r_1, r_2)$ is often vague or diffuse, we employ a diffuse prior for θ'. Assume that the priors of the change-point m, of $\phi^{(p)}$ and of $\psi^{(p)}$ are given by

$$\begin{aligned} \pi(m) &\propto \frac{1}{n-2}; \quad m = 1, \ldots, n-2, \\ \pi(\phi^{(p)}) &\propto \text{constant in} \quad \Phi^{(p)}, \\ \pi(\psi^{(p)}) &\propto \text{constant in} \quad \Phi^{(p)}, \end{aligned}$$

where $\Phi^{(p)} = \Phi_1^{(p)} \cap \Phi_2^{(p)}$.

The parameters m, $\phi^{(p)}$ and θ' being assumed independent. The prior distribution of θ is, therefore, stated as

$$\pi(\theta) \propto \frac{1}{r_1 r_2}, \tag{3}$$

where $m = 1, \ldots, n-2$, $\mu_1, \mu_2 \in \mathrm{R}$, $r_1, r_2 \in \mathrm{R}_+^*$ and $\phi_i, \psi_i \in \Phi^{(p)}$ for $i = 1, \ldots, p$. The posterior distribution of θ, obtained by combination of Equations (2) and (3) is formulated as

$$\pi(\theta|y) \propto r_1^{\frac{m}{2}-1} r_2^{\frac{n-m}{2}-1} \exp\left\{-\frac{r_1}{2}\left[\sum_{t=1}^{m}(y_t - \mu_1 - \sum_{i=1}^{p}\phi_i(y_{t-i} - \mu_1)\right]^2\right\}$$
$$\exp\left\{-\frac{r_2}{2}\left[\sum_{t=m+1}^{m+p}(y_t - \mu_2 - \sum_{i=1}^{p}\psi_i\Big(y_{t-i} - \gamma_{t-i}\mu_1 - (1-\gamma_{t-i})\mu_2\Big)\right]^2\right\}$$
$$\exp\left\{-\frac{r_2}{2}\left[\sum_{t=m+p+1}^{n}(y_t - \mu_2 - \sum_{i=1}^{p}\psi_i(y_{t-i} - \mu_2)\right]^2\right\}.$$

In the following, we give the joint posterior distribution of the parameter $\Phi = (m, \mu_1, \delta, \phi^{(p)}, \rho^{(p)}, \tau)$. By transforming the parameter set $\Theta = (m, \mu_1, \mu_2, \phi^{(p)}, \psi^{(p)}, r_1, r_2)$ into $\Phi = (m, \mu_1, \delta, \phi^{(p)}, \rho^{(p)}, \tau)$, we can form the joint posterior distribution of Φ, that is, we have

$$\pi(\Phi \mid y) = \int_{r_2} \pi(m, \mu_1, \delta + \mu_1, \phi_1, \rho^{(p)} + \phi^{(p)}, r_2\tau, r_2/y) \mid r_2 \mid \mathrm{d}r_2, \tag{4}$$
$$\tau^{\frac{m}{2}-1}\left\{\tau\sum_{t=1}^{m}\left(y_t - \mu_1 - \sum_{i=1}^{p}\phi_i(y_{t-i} - \mu_1)\right)^2\right.$$
$$+\sum_{t=m+1}^{m+p}\left(y_t - \delta - \mu_1 - \sum_{1}^{p}(\rho_i + \phi_i)(y_{t-i} - \gamma_{t-i}\mu_1 - (1-\gamma_{t-i})(\delta + \mu_1))\right)^2$$
$$\left.+\sum_{t=m+p+1}^{n}\left(y_t - \delta - \mu_1 - \sum_{1}^{p}(\rho_i + \phi_i)(y_{t-i} - \delta - \mu_1)\right)^2\right\}^{-\frac{n}{2}}.$$

The posterior conditional distribution of δ is stated as follows. Equation (4) can be written as

$$\pi(\Phi \mid y) \propto \tau^{\frac{m}{2}-1}\left\{\tau SS_1(m, \mu_1, \phi^{(p)}) + SS_2(m, \mu_1, \phi^{(p)}, \rho^{(p)}) + \Lambda_1\left(\delta - \widehat{\delta}\left(m, \mu_1, \phi^{(p)}, \rho^{(p)}\right)\right)^2\right\}^{-\frac{n}{2}}, \tag{5}$$

where

$$\Lambda_1 = \sum_{m+1}^{m+p}\left(1 - \sum_{1}^{p}(1-\gamma_{t-i})(\rho_i + \phi_i)\right)^2 + (n-m-p)\left(1 - \sum_{1}^{p}(\rho_i + \phi_i)\right)^2,$$

$\widehat{\delta}(m, \mu_1, \phi^{(p)}, \rho^{(p)}) = \Lambda_2/\Lambda_1$, with

$$\Lambda_2 = \sum_{m+1}^{m+p} (1-(1-\gamma_{t-i})(\rho_i+\phi_i)) \left(y_t - \mu_1 - \sum_{i=1}^{p} (\rho_i+\phi_i)(y_{t-i}-\mu_1 \right) + \left(1 - \sum_{i=1}^{p}(\rho_i+\phi_i)\right) \left(\sum_{m+p+1}^{n} (y_t - \mu_1 - \sum_{i=1}^{p}(\rho_i+\phi_i)(y_{t-i}-\mu_1) \right),$$

and

$$SS_1(m, \mu_1, \phi^{(p)}) = \sum_{t=1}^{m} \left(y_t - \mu_1 - \sum_{i=1}^{p} \phi_i (y_{t-i} - \mu_1) \right)^2, \tag{6}$$

$$SS_2(m, \mu_1, \phi^{(p)}, \rho^{(p)}) = \sum_{t=m+1}^{n} \left(y_t - \mu_1 - \sum_{1}^{p} (\rho_i+\phi_i)(y_{t-i} - \mu_) \right)^2 - \frac{\Lambda_2^2}{\Lambda_1}. \tag{7}$$

Following the Bayes theorem, the conditional posterior distribution of δ is given by

$$\pi(\delta | m, \mu_1, \phi^{(p)}, \rho^{(p)}, \tau, y) \propto \left\{ 1 + \frac{(\delta - \widehat{\delta}\left(m, \mu_1, \phi^{(p)}, \rho^{(p)}\right))^2}{(n-1) S_1^2(m, \mu_1, \phi^{(p)}, \rho^{(p)}, \tau)} \right\}^{-\frac{n}{2}},$$

where $S_1^2(m, \mu_1, \phi^{(p)}, \rho^{(p)}, \tau) = (\tau SS_1(m, \mu_1, \phi^{(p)}) + SS_2(m, \mu_1, \phi^{(p)}, \rho^{(p)}))/((n-1)\Lambda_1)$. Given m, μ_1, $\phi^{(p)}$, $\rho^{(p)}$ and τ, the conditional posterior distribution of δ is distributed as a Student-t distribution with location parameter $\widehat{\delta}(m, \mu_1, \phi^{(p)}, \rho^{(p)})$, precision $S_1^2(m, \mu_1, \phi^{(p)}, \rho^{(p)}, \tau)$ and $(n-1)$ degrees of freedom. Equivalently, the quantity

$$T(\delta) = \frac{\delta - \widehat{\delta}\left(m, \mu_1, \phi^{(p)}, \rho^{(p)}\right)}{S_1\left(m, \mu_1, \phi^{(p)}, \rho^{(p)}, \tau\right)},$$

is distributed a posteriori as a conditional Student-t distribution with $(n-1)$ degrees of freedom.

The posterior conditional distributions of ρ_j is formulated as follows. Equation (4) can also be written as

$$\pi(\Phi \mid y) \propto \tau^{\frac{m}{2}-1} \left\{ \Lambda_{5j} - \frac{\Lambda_{4j}^2}{\Lambda_{3j}} + \Lambda_{3j} \left(\rho_j - \widehat{\rho}_j \left(m, \mu_1, \delta, \phi^{(p)}, \rho^{(-j)} \right) \right)^2 \right\}^{-\frac{n}{2}},$$

where, $\widehat{\rho}_j\left(m, \mu_1, \delta, \phi^{(p)}, \rho^{(-j)}\right) = \Lambda_{4j}/\Lambda_{3j}$, with

$$\Lambda_{3j} = \sum_{t=m+1}^{m+p} (y_{t-j} - \gamma_{t-j}\mu_1 - (1-\gamma_{t-j})(\delta+\mu_1))^2 + \sum_{t=m+p+1}^{n} (y_{t-j} - \delta - \mu_1)^2;$$

$$\begin{aligned}
\Lambda_{4j} = & \sum_{m+1}^{m+p} \Big[y_{t-j} - \gamma_{t-j}\mu_1 - (1-\gamma_{t-j})(\delta+\mu_1)\Big] \\
& \Big[y_t - \delta - \mu_1 - \sum_{i=1}^{p} \phi_i(y_{t-i} - \gamma_{t-i}\mu_1 - (1-\gamma_{t-i})(\delta+\mu_1)) \\
& - \sum_{i\neq j}^{p} \rho_i(y_{t-i} - \gamma_{t-i}\mu_1 - (1-\gamma_{t-i})(\delta+\mu_1))\Big] \\
& \sum_{m+p+1}^{n} [y_{t-j} - \delta - \mu_1]) \Big[y_t - \delta - \mu_1 - \sum_{i=1}^{p} \phi_i(y_{t-i} - \delta - \mu_1) - \sum_{i\neq j}^{p} \rho_i(y_{t-i} - \delta - \mu_1)\Big];
\end{aligned}$$

$$\begin{aligned}
\Lambda_{5j} = & \tau \sum_{t=1}^{m} \Big(y_t - \mu_1 - \sum_{i=1}^{p} \phi_i(y_{t-i} - \mu_1)\Big)^2 \\
& + \sum_{m+1}^{m+p} \Big(y_t - \delta - \mu_1 - \sum_{1}^{p} \phi_i(y_{t-i} - \gamma_{t-i}\mu_1 - (1-\gamma_{t-i})(\delta+\mu_1)) \\
& - \sum_{i\neq j}^{p} \rho_i(y_{t-i} - \gamma_{t-i}\mu_1 - (1-\gamma_{t-i})(\delta+\mu_1))\Big)^2 \\
& + \sum_{m+p+1}^{n} \Big(y_t - \delta - \mu_1 - \sum_{i=1}^{p} \phi_i(y_{t-i} - \delta - \mu_1) - \sum_{i\neq j}^{p} \rho_i(y_{t-i} - \delta - \mu_1)\Big)^2.
\end{aligned}$$

Following the Bayes theorem, the posterior conditional distribution of ρ_j, for $j = 1, \ldots, p$, is given by

$$\pi(\rho_j | m, \mu_1, \phi^{(p)}, \delta, \rho^{(-j)}, \tau, y) \propto \left\{1 + \frac{(\rho_j - \widehat{\rho}_j(m, \phi^{(p)}, \mu_1, \delta, \rho^{(-j)})^2}{(n-1)S_{2j}^2(m, \phi^{(p)}, \mu_1, \delta, \rho^{(-j)}, \tau)}\right\}^{-\frac{n}{2}},$$

where

$$S_{2j}^2(m, \phi^{(p)}, \mu_1, \delta, \rho^{(-j)}, \tau) = \frac{\Lambda_{5j} - \dfrac{\Lambda_{4j}^2}{\Lambda_{3j}}}{(n-1)\Lambda_{3j}}.$$

For $j = 1, \ldots, p$, given m, μ_1, $\phi^{(p)}$, δ, $\rho^{(-j)}$ and τ, the conditional posterior distribution of ρ_j is distributed as a Student-t distribution with location parameter $\widehat{\rho}_j(m, \phi^{(p)}, \mu_1, \delta, \rho^{(-j)})$, precision $S_{2j}(m, \phi^{(p)}, \mu_1, \delta, \rho^{(-j)}, \tau)$ and $(n-1)$ degrees of freedom. Thereby, the quantity,

$$S_j(\rho_j) = \frac{\rho_j - \widehat{\rho}_j(m, \phi^p, \mu_1, \delta, \rho^{(-j)})}{S_2(m, \phi^{(p)}, \mu_1, \delta, \rho^{(-j)}, \tau)},$$

is distributed a posteriori as a conditional Student-t distribution with $(n-1)$ degrees of freedom.

The posterior conditional distributions of τ is expressed as follows. The integration of Equation (5) with respect to δ gives the joint posterior distribution of m, μ_1, $\phi^{(p)}$, $\rho^{(p)}$ and τ by

$$\pi(m,\mu_1,\phi^{(p)},\rho^{(p)},\tau|y) \propto \tau^{\frac{m}{2}-1}\Lambda_1^{-1/2}\left\{\tau SS_1\left(m,\mu_1,\phi^{(p)}\right)+SS_2\left(m,\mu_1,\phi^{(p)},\rho^{(p)}\right)\right\}^{-\frac{(n-1)}{2}}, \tag{8}$$

by application of the Bayes theorem, the conditional posterior distributions of τ is given by

$$\pi\left(\tau|m,\mu_1,\phi_1,\rho,y\right) \propto \tau^{\frac{m}{2}-1}\left\{\tau SS_1(m,\mu_1,\phi^{(p)})+SS_2(m,\mu_1,\phi^{(p)},\rho^{(p)})\right\}^{-\frac{(n-1)}{2}},$$

where $SS_1(m,\mu_1,\phi^{(p)})$ and $SS_2(m,\mu_1,\phi^{(p),\rho^{(p)}})$ are given in Equations (6) and (7), respectively. Given m, μ_1, $\phi^{(p)}$, $\rho^{(p)}$, the quantity

$$F\left(\tau\right)=\tau\frac{SS_1\left(m,\mu_1,\phi^{(p)}\right)/m}{SS_2(m,\mu_1,\phi^{(p)},\rho^{(p)})/\left(n-m-1\right)},$$

is distributed a posteriori as a conditional F distribution with $(m,n-m-1)$ degrees of freedom.

The posterior conditional distribution of ϕ_j, for $j=1,\ldots,p$, is considered as follows. Still, the formula in Equation (4) can be written as

$$\pi(\Phi \mid y) \propto \tau^{\frac{m}{2}-1}\left\{\Lambda_{8j}-\frac{\Lambda_{7j}^2}{\Lambda_{6j}}+\Lambda_6\left(\phi_j-\widehat{\phi}_j\left(m,\mu_1,\delta,\phi^{(-j)},\rho^{(p)}\right)\right)^2\right\}^{-\frac{n}{2}},$$

where

$$\begin{aligned}
\Lambda_{6j} &= \tau\sum_1^m(y_{t-j}-\mu_1)^2+\sum_{m+1}^{m+p}\left(y_{t-j}-\gamma_{t-j}\mu_1-(1-\gamma_{t-j})(\delta+\mu_1)\right)^2\\
&\quad+\sum_{m+p+1}^{n}(y_{t-j}-\delta-\mu_1)^2;\\
\Lambda_{7j} &= \tau\sum_1^m(y_{t-j}-\mu_1)(y_t-\mu_1-\sum_{i\neq j}\phi_i(y_{t-i}-\mu_1))\\
&\quad+\sum_{m+1}^{m+p}\left(y_{t-j}-\gamma_{t-j}\mu_1-(1-\gamma_{t-j})(\delta+\mu_1)\right)\\
&\quad\Big(y_t-\delta-\mu_1-\sum_{i=1}^{p}\rho_i(y_{t-i}-\gamma_{t-i}\mu_1-(1-\gamma_{t-i})(\delta+\mu_1))\\
&\quad-\sum_{i\neq j}\phi_i(y_{t-i}-\gamma_{t-i}\mu_1-(1-\gamma_{t-i})(\delta+\mu_1))\Big)\\
&\quad+\sum_{m+p+1}^{n}(y_{t-j}-\delta-\mu_1)(y_t-\delta-\mu_1-\sum_{i=1}^{p}\rho_i(y_{t-i}-\delta-\mu_1)-\sum_{i\neq j}\phi_i(y_{t-i}-\delta-\mu_1));
\end{aligned}$$

$$
\begin{aligned}
\Lambda_{8j} = \tau \sum_{1}^{m} (y_t - \mu_1 - \sum_{i \neq j} \phi_i (y_{t-i} - \mu_1)^2 \\
+ \sum_{m+1}^{m+p} \Big(y_t - \delta - \mu_1 - \sum_{i=1}^{p} \rho_i (y_{t-i} - \gamma_{t-i}\mu_1 - (1-\gamma_{t-i})(\delta + \mu_1)) \\
- \sum_{i \neq j} \phi_i (y_{t-i} - \gamma_{t-i}\mu_1 - (1-\gamma_{t-i})(\delta + \mu_1)) \Big)^2 \\
+ \sum_{m+p+1}^{n} \Big(y_t - \delta - \mu_1 - \sum_{i=1}^{p} \rho_i (y_{t-i} - \delta - \mu_1) - \sum_{i \neq j} \phi_i (y_{t-i} - \delta - \mu_1) \Big)^2;
\end{aligned}
$$

and $\widehat{\phi}_j(m, \mu_1, \delta, \phi^{(j)}, \rho^{(p)}) = \Lambda_7/\Lambda_6$. Following the Bayes theorem, the posterior conditional distribution of ϕ_j, for $j = 1, \ldots, p$, is given by

$$
\pi(\phi_j | m, \rho^{(p)}, \mu_1, \phi^{(-j)}, \delta, \tau, y) \propto \left\{ 1 + \frac{(\phi_j - \widehat{\phi}_j(m, \rho^{(p)}, \mu_1, \phi^{(-j)}, \delta, \tau))^2}{(n-1) S_{3j}^2 (m, \rho^{(p)}, \mu_1, \phi^{(-j)}, \delta, \tau)} \right\}^{-\dfrac{n}{2}},
$$

where $S_{3j}^2(m, \rho^{(p)}, \mu_1, \phi^{(-j)}, \delta, \tau) = (\Lambda_{8j} - \Lambda_{7j}^2/\Lambda_{6j})/((n-1)\Lambda_{6j})$. For $j = 1, \ldots, p$, given m, μ_1, $\phi^{(-j)}$, $\rho^{(p)}$, δ, and τ, the conditional posterior distribution of ϕ_j is distributed as a Student-t distribution with location parameter $\widehat{\phi}_j(m, \rho^{(p)}, \mu_1, \phi^{(-j)}, \delta, \tau)$, precision $S_{3j}(m, \rho^{(p)}, \mu_1, \phi^{(-j)}, \delta, \tau)$ and $(n-1)$ degrees of freedom.

The posterior conditional distribution of μ_1 is given next. We can write Equation (4) as

$$
\pi(\Phi \mid y) \propto \tau^{\frac{m}{2}-1} \left\{ \Lambda_{11} - \frac{\Lambda_{10}^2}{\Lambda_9} + \Lambda_9 \left(\mu_1 - \widehat{\mu}_1 \left(m, \phi^{(p)}, \rho^{(p)}, \delta, \tau \right) \right)^2 \right\}^{-\frac{n}{2}},
$$

where

$$
\begin{aligned}
\Lambda_9 &= m\tau \left(1 - \sum_{i=1}^{p} \phi_i \right)^2 + (n-m) \left(1 - \sum_{i=1}^{p} (\rho_i + \phi_i) \right)^2; \\
\Lambda_{10} &= \tau (1 - \sum_{i=1}^{p} \phi_i) \sum_{1}^{m} \left(y_t - \sum_{i=1}^{p} \phi_i y_{t-i} \right) \\
&\quad + (1 - \sum_{i=1}^{p} (\rho_i + \phi_i) \sum_{m+1}^{m+p} \left(y_t - \delta - \sum_{i=1}^{p} (\rho_i + \phi_i)(y_{t-i} - (1-\gamma_{t-i})\delta \right) \\
&\quad (1 - \sum_{i=1}^{p} (\rho_i + \phi_i)) \sum_{m+p+1}^{n} \left(y_t - \delta - \sum_{i=1}^{p} (\rho_i + \phi_i)(y_{t-i} - \delta) \right); \\
\Lambda_{11} &= \tau \sum_{1}^{m} \left(y_t - \sum_{i=1}^{p} \phi_i y_{t-i} \right)^2 + \sum_{m+1}^{m+p} \left(y_t - \delta - \sum_{i=1}^{p} (\rho_i + \phi_i)(y_{t-i} - (1-\gamma_{t-i})\delta) \right)^2 \\
&\quad \sum_{m+p+1}^{n} \left(y_t - \delta - \sum_{i=1}^{p} (\rho + \phi_i)(y_{t-i} - \delta) \right)^2;
\end{aligned}
$$

and $\widehat{\mu}_1(m, \phi^{(p)}, \rho^{(p)}, \delta, \tau) = \Lambda_{10}/\Lambda_9$. By the Bayes theorem, the posterior conditional distribution of μ_1 is given by

$$\pi(\mu_1|m, \phi^{(p)}, \rho^{(p)}, \delta, \tau, y) \propto \left\{ 1 + \frac{(\mu_1 - \widehat{\mu}_1(m, \phi^{(p)}, \rho^{(p)}, \delta, \tau))^2}{(n-1)S_4^2(m, \rho^{(p)}, \phi^{(p)}, \delta, \tau)} \right\}^{-\frac{n}{2}},$$

where $S_4^2(m, \rho^{(p)}, \phi^{(p)}, \delta, \tau) = (\Lambda_{11} - \Lambda_{10}^2/\Lambda_9)/((n-1)\Lambda_9)$. Given m, $\phi^{(p)}$, $\rho^{(p)}$, δ and τ, the conditional posterior distribution of μ_1 is distributed as a Student-t distribution with location parameter $\widehat{\mu}_1(m, \rho^{(p)}, \phi^{(p)}, \delta, \tau)$, precision $S_4(m, \rho^{(p)}, \phi^{(p)}, \delta, \tau)$ and $(n-1)$ degrees of freedom.

The posterior conditional distributions of m is stated next. From the joint posterior distribution of m, μ_1, $\phi^{(p)}$, $\rho^{(p)}$ and τ given in Equation (8), the conditional posterior distributions of m is given by

$$\pi(m|\mu_1, \phi^{(p)}, \rho^{(p)}, \tau, y) \propto \tau^{\frac{m}{2}-1} \Lambda_1^{-1/2} \left\{ \tau SS_1\left(m, \mu_1, \phi^{(p)}\right) + SS_2\left(m, \mu_1, \phi^{(p)}, \rho^{(p)}\right) \right\}^{-\frac{(n-1)}{2}},$$

where $SS_1(m, \mu_1, \phi^{(p)})$ and $SS_2(m, \mu_1, \phi^{(p),\rho^{(p)}})$ are given in Equations (6) and (7), respectively.

Remark 1 As the degrees of freedom m and $n-m-1$ of F distribution are greater or equal to 1, this implies that the change point m belongs to $\{1, n-2\}$.

The unconditional posterior distributions of $T(\delta)$, $S_j(\rho_j)$, for $j = 1, \dots, p$, and $F(\tau)$ are given, respectively, by

$$\pi(T(\delta)|y) = \sum_m \int_\tau \int_{\rho^{(p)}} \int_{\phi^{(p)}} \int_{\mu_1} \pi(T(\delta)|m, \mu_1, \phi^{(p)}, \rho^{(p)}, \tau, y) \pi(\mu_1|m, \phi^{(p)}, \rho^{(p)}, \tau, y) \\ \pi(\phi^{(p)}|m, \rho, \phi^{(p)}, \tau, y) \pi(\rho^{(p)}|m, \rho^{(p-1)}, \tau, y) \pi(\tau|m, y) \pi(m|y) \mathrm{d}\mu_1 \mathrm{d}\phi^{(p)} \mathrm{d}\rho^{(p)} \mathrm{d}\tau, \quad (9)$$

$$\pi(S_j(\rho_j)|y) = \sum_m \int_\tau \int_\delta \int_{\rho^{(j)}} \int_{\phi^{(p)}} \int_{\mu_1} \pi(S_j(\rho_j)|m, \mu_1, \phi^{(p)}, \delta, \tau, y) \pi(\mu_1|m, \phi^{(p)}, \delta, \tau, y) \\ \pi(\phi^{(p)}|m, \delta, \tau, y) \pi(\rho^{(-j)}|m, \rho^{(p-j)}, \delta, \tau, m) \pi(\delta|m, \tau, y) \\ \pi(\tau|m, y) \pi(m|y) \mathrm{d}\mu_1 \mathrm{d}\phi^{(p)} \mathrm{d}\rho^{(-j)} \mathrm{d}\delta \mathrm{d}\tau, j = 1, \dots, p, \quad (10)$$

$$\pi(F(\tau)|y) = \sum_m \int_\delta \int_{\rho^{(p)}} \int_{\phi^{(p)}} \int_{\mu_1} \pi(F(\tau)|m, \mu_1, \phi^{(p)}, \rho^{(p)}, \delta, y) \pi(\mu_1|m, \phi^{(p)}, \rho^{(p)}, \delta, y) \\ \pi(\phi^{(p)}|m, \rho^{(p)}, \delta, y) \pi(\delta|m, \rho^{(p)}, y) \pi(\rho^{(p)}|m, \rho^{(p-j)}, y) \pi(m|y) \mathrm{d}\mu_1 \mathrm{d}\phi^{(p)} \mathrm{d}\rho^{(p)} \mathrm{d}\delta, \quad (11)$$

where

$$\pi(\rho^{(p)}|\beta, \rho^{(p-1)}, y) = \pi(\rho_1|\beta, \rho_2, \dots, \rho_p, y) \pi(\rho_2|\beta, \rho_1, \rho_3, \dots, \rho_p, y), \dots, \pi(\rho_p|\beta, \rho_1, \dots, \rho_{p-1}, y),$$

and $\pi(\rho^{(-j)}|\beta, \rho^{(p-j)}, y) =$

$$\begin{cases} \pi(\rho_2|\beta,\rho_3,\ldots,\rho_p,y)\pi(\rho_3|\beta,\rho_2,\rho_4,\ldots,\rho_p,y),\ldots,\pi(\rho_p|\beta,\rho_2,\ldots,\rho_{p-1},y), & j=1;\\ \pi(\rho_1|\beta,\rho_3,\ldots,\rho_p,y)\pi(\rho_3|\beta,\rho_1,\rho_4,\ldots,\rho_p,y)\ldots\pi(\rho_p|\beta,\rho_1,\rho_3,\ldots,\rho_{p-1},y), & j=2;\\ \vdots & \vdots\\ \pi(\rho_1|\beta,\rho_2,\ldots,\rho_{p-1},y)\pi(\rho_2|\beta,\rho_1,\rho_3,\ldots,\rho_{p-1},y),\ldots,\pi(\rho_{p-1}|\beta,\rho_1,\rho_2,\ldots,\rho_{p-2},y), & j=p. \end{cases}$$

The null hypothesis H_0 can be divided into $p+2$ sub-hypotheses H_{01}: $\delta = \mu_2 - \mu_1 = 0$, H_{02j}: $\rho_j = \phi_j - \psi_j = 0$, and H_{03}: $\tau = \sigma_2^2/\sigma_1^2 = 1$, and H_0 could be rejected if either of these $p+2$ sub-hypotheses is rejected. The separation of the null into several sub-hypotheses would be helpful to determine which parameters have been changed at time m. One defines separately the HPD credible sets of $T(\delta)$, $S_j(\rho_j)$ and $F(\tau)$ based on conditional distributions. The credible set are used to define the unconditional p-value and thereby an unconditional test, the bayesian significance test of change in the parameters of autoregressive time series.

Given m, μ_1, $\phi^{(p)}$, $\rho^{(p)}$ and τ the $(1-\alpha)$-credible set for $T(\delta)$ is defined as

$$C_\delta = \Big\{T(\delta) \mid T(\delta)| < t_{\alpha/2}(n-1)\Big\},$$

where $t_{\alpha/2}(n-1))$ is the $100(1-\alpha/2)$th quantile of a Student-t distribution with $(n-1)$ degrees of freedom. Hence, given m, μ_1, $\phi^{(p)}$, $\rho^{(p)}$ and τ the decision rule for H_{01} is to reject if $T(0) \in \overline{C_\delta}$, where $\overline{C_\delta}$ is the complement of C_δ.

The unconditional p-value of the hypothesis H_{01} calculated from Equation (9) yields

$$\begin{aligned} P_{\delta=0|y} &= 2\sum_m \int_\tau \int_{\rho^{(p)}} \int_{\phi^{(p)}} \int_{\mu_1} \{1 - \mathcal{T}_{n-1}(|\, T(0)\,|)\} \\ &\quad \pi(m,\mu_1,\phi^{(p)},\delta,\rho^{(p)},\tau|y)\mathrm{d}\mu_1\mathrm{d}\phi^{(p)}\mathrm{d}\rho^{(p)}\mathrm{d}\tau, \\ &= 2E_mE_\tau E_{\rho^{(p)}}E_{\mu_1}E_{\phi^{(p)}}\{1 - \mathcal{T}_{n-1}(|\, t(0)\,|)\}, \end{aligned} \tag{12}$$

The sub-hypothesis H_{01} is rejected unconditionally at α significance level if $P_{\delta=0|y} < \alpha$.

The unconditional p-value of the hypothesis H_{02j}, for $j = 1, \ldots, p$, calculated from Equation (10), is given by

$$\begin{aligned} P_{\rho_j=0|y} &= 2\sum_m \int_\tau \int_{\mu_1} \int_\delta \int_{\rho^{(-j)}} \int_{\phi^{(p)}} \int_{\mu_1} \{1 - \mathcal{T}_{n-1}(|\, S_j(0)\,|)\} \\ &\quad \pi(m,\mu_1,\phi^{(p)},\delta,\rho^{(-j)},\tau|y)\mathrm{d}\mu_1\mathrm{d}\phi^{(p)}\mathrm{d}\rho^{(-j)}\mathrm{d}\delta\mathrm{d}\tau, \\ &= 2E_mE_\tau E_{\mu_1}E_\delta E_{\rho^{(-j)}}E_{\phi^{(p)}}\{1 - \mathcal{T}_{n-1}(|\, t(0)\,|)\}, \end{aligned} \tag{13}$$

where $\mathcal{T}_{n-1}$ is the cumulative distribution function of the Student-t distribution with $(n-1)$ degrees of freedom. For $j = 1, \ldots, p$, the sub-hypothesis H_{02j} is rejected unconditionally at α significance level if $P_{\rho_j=0|y} < \alpha$. Where, the sub-hypothesis H_{02} is rejected unconditionally at α significance level if

$$P_{\rho=0|y} := \min_{1\leq j\leq p}\{P_{\rho_j=0}|y\} < \alpha.$$

Likewise, the unconditional p-value of H_{03} calculated from Equation (11) is stated as

$$\begin{aligned} P_{\tau=1|y} &= 2\sum_{m}\int_{\rho^{(p)}}\int_{\phi^{(p)}}\int_{\mu_1}\left\{1-\mathcal{F}_{m,n-m-1}[\max\left(F(1),1/F(1)\right)]\right\} \\ &\quad \pi\left(m,\mu_1,\phi^{(p)},\rho^{(p)}|y\right)\mathrm{d}\mu_1\mathrm{d}\phi_1\mathrm{d}\rho, \\ &= 2E_m E_{\rho^{(p)}}E_{\phi^{(p)}}E_{\mu_1}\left\{1-\mathcal{F}_{m,n-m-1}[\max\left(F(1),1/F(1)\right)]\right\}, \end{aligned} \tag{14}$$

where $\mathcal{F}_{m,n-m-1}$ is the cumulative distribution function of the Fisher F distribution with $(m, n-m-1)$ degrees of freedom. The sub-hypothesis H_{03} is rejected unconditionally at α significance level if $P_{\tau=1|y} < \alpha$. Therefore, the null hypothesis H_0 well be rejected unconditionally at α significance level if $\min\{P_{\delta=0|y}, P_{\rho=0|y}, P_{\tau=0|y}\} < \alpha$, and thus define the bayesian significance test of change in the parameters of autoregressive time series AR(p) of known order p. The test allows to test the change in the $p+2$ parameters of the AR(p) model in an individual way.

The notations E_{μ_1}, $E_{\phi^{(p)}}$, $E_{\rho^{(p)}}$, $E_{\rho^{(-j)}}$, E_δ, E_τ and E_m are the expectations taken with respect to μ_1, $\phi^{(p)}$, $\rho^{(p)}$, $\rho^{(-j)}$, δ, τ, and m, respectively.

The quantities given in Equations (12), (13) and (14) are evaluated numerically by the Gibbs sampler algorithm using the conditional posterior distributions given in Section 3.

The Gibbs sampler was introduced by Geman and Geman (1984) as a way of simulating from high-dimensional complex distributions arising in image restoration, is a Markovian updating scheme enabling one to obtain samples from a joint distribution via iterated sampling from full conditional distributions. Although most applications of Gibbs sampler have been in Bayesian models, it is also extremely useful in classical (likelihood) calculations Casella and George (1992). In Bayesian framework, the common objective is to produce posterior densities for, or estimate of, parameters of interest. The algorithm is also very useful for the calculation of high dimensional integrals. Therefore, the use of Gibbs sampler algorithm allows us to reduce in a huge way the calculation of complex high-dimensional integration in Equations (12), (13) and (14). Detailed investigation of the Gibbs sampler applied to general Bayesian calculation is given by Gelfand and Smith (1990), Gelfand et al. (1990) and Gelfand (2000).

4. Simulation results

In this section we conduct a set of controlled simulation studies to evaluate the performance of the proposed test presented in Section 3. We simulated a sample from the model given in Equation (1) with $p = 1$, $n = 200$, $m = 100$, $\mu_1 = 0.0$, $\mu_2 = 0.5$, $\phi_1 = 0.3$, $\phi_2 = -0.2$, $\sigma_1^2 = 1.0$ and $\sigma_2^2 = 0.5$. The assumed values for y_0 is 1. From these observations, by the application of the Gibbs sampler algorithm with 10,000 repetitions, we approximate the posterior density of the change point m, the posterior density of δ, the posterior density of ρ, of the variance ratio τ and the unconditional p-values for the hypothesis H_{01}: $\delta = 0$, H_{012}: $\rho = 0$ and H_{03}: $\tau = 1$. The results are given in Tables 1-3.

Tables 1 and 2 list the posterior density of the change point at values around the true value of m and the unconditional p-values for the sub-null-hypotheses H_{01}, H_{02} and H_{03}. From Table 1, we can readily see that the posterior mode is equal to the true value of the change point m. Based on the unconditional p-values given in Table 2 the no change in δ, ρ and τ is obviously rejected at 1% significance levels, respectively.

Tables 3 summarize the posterior estimates of the parameters m, δ, ρ and τ. The estimates for the parameters of the series in Figure 1 are generally close to the true values. Also, we

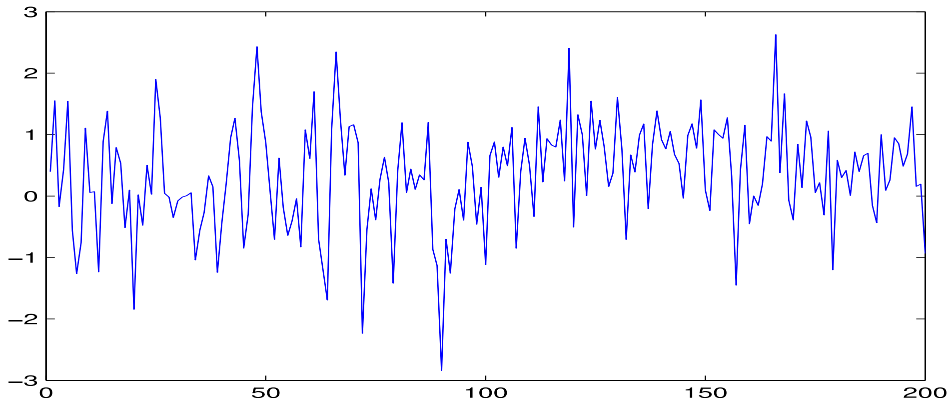


Figure 1. Simulated observations y_t.

Table 1. The posterior density of m.

m	$\pi(m\|y)$	m	$\pi(m\|y)$
89	0.0000	100	0.2321
90	0.0017	101	0.1396
91	0.0040	102	0.0782
92	0.0304	103	0.0541
93	0.0404	104	0.0311
94	0.0368	105	0.0192
95	0.0551	106	0.0120
96	0.0313	107	0.0378
97	0.0214	108	0.0274
98	0.0363	109	0.0154
99	0.0368	110	0.0107

Table 2. The unconditional p-values of the hypothesis H_{01}, H_{02} and H_{03}.

Sub-null-hypothesis	H_{01}	H_{02}	H_{03}
p-values	4.8452×10^{-5}	0.0027	0.0017

clearly see that, all the 95% HPD sets of the parameters contain the true value of all the parameters.

Table 3. Posterior estimates of the parameters m, δ, ρ and τ.

Parameters	True values	Median	Mean (SD)	2.5%	97,5%
m	100	100	100.56(4.9418)	92	111
$\delta = \mu_2 - \mu_1$	0.5	0.4872	0.4869(0.1115)	0.2709	0.7098
$\rho = \phi_2 - \phi_1$	−0.5	−0.4230	−0.4239(0.1368)	−0.6897	−0.1515
$\tau = \sigma_2^2/\sigma_1^2$	0.5	0.5023	0.5138(0.1129)	0.3291	0.7649

Figures 2-5 give the posterior distribution of the parameters m, δ, ρ and τ. They indicate that the posterior mode is around the true values of the parameters. Thus, an estimate of the true values of the parameters is given by the posterior mode of the respective posterior distributions.

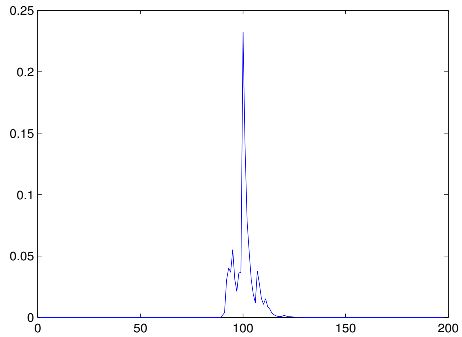


Figure 2. Posterior density function of the change point m.

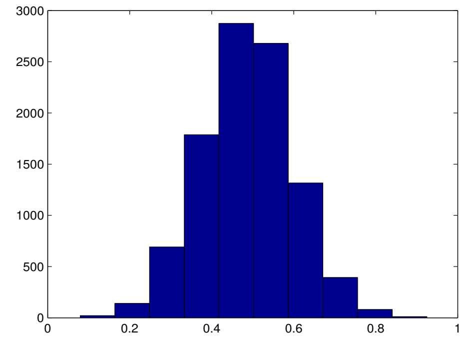


Figure 3. Histogram of posterior distribution of the parameter δ.

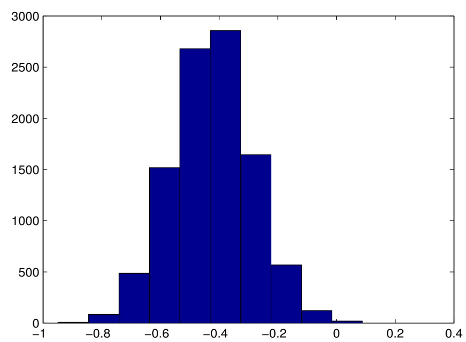


Figure 4. Histogram of posterior distribution of the parameter ρ.

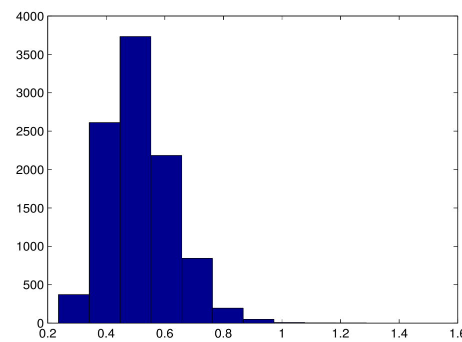


Figure 5. Histogram of posterior distribution of the parameter τ.

Furthermore, Table 4 presents the unconditional p-values of the sub-null-hypotheses H_{01}, H_{02} and H_{03} for $n = 200$, $m = 100$ and different values of the parameters μ_1, μ_2, ϕ_1, ϕ_2, σ_1^2 and σ_2^2. Several cases are considered, stability in one of the parameters and change in the other two and in the last case the three parameters are stable (without change). The results show that the p-values of sub-hypotheses corresponding to the stable parameters do not allow to reject the corresponding sub-hypothesis. While, for the other sub-hypotheses where the parameters exhibiting changes, the corresponding p values make it possible to reject these sub-hypotheses at 5% significance level. For example, with $\mu_1 = 0.0$, $\mu_2 = 0.5$, $\phi_1 = 0.3$, $\phi_2 = 0.3$ and $\sigma_1^2 = 1.0$, $\sigma_2^2 = 0.5$, the p-values $P_{\delta=0|y}$ and $P_{\tau=1|y}$ are respectively 0.0187 and 0.0146. Thus, the sub-hypotheses H_{01} and H_{03} are rejected at 5% significance level. The p-value $P_{\rho=0|y}$ is 0.4134, therefore, the sub-hypothesis H_{02} cannot be rejected. Note that the parameter ϕ is stable, that is, $\rho = \phi_2 - \phi_1 = 0$.

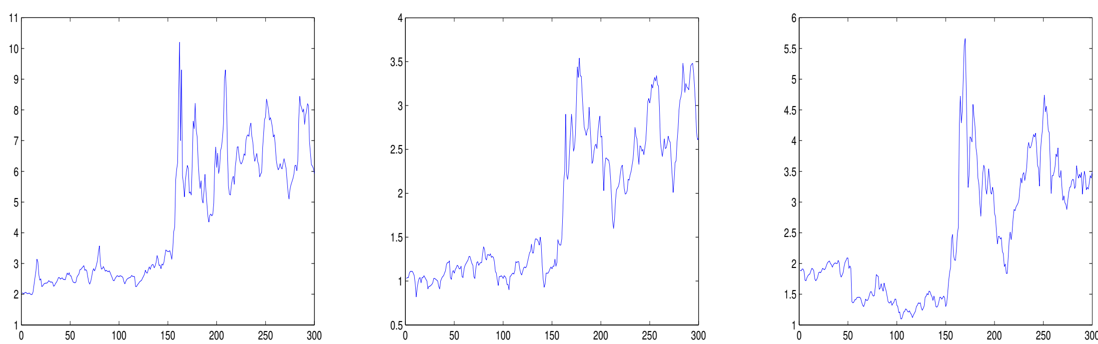
To study the performance of the Bayesian significance test for detecting structural changes in the parameters of autoregressive AR(p), we simulated 1000 samples from the model given in Equation (1) with $p = 1$ and different values of n, m, μ_1, μ_2, ϕ_1, ϕ_2, σ_1^2 and σ_2^2 and we computed the rejection rates (the number of times the hypothesis is rejected divided by the total number of samples) of sub-hypotheses H_{01}, H_{02} and H_{03} at 5% significance level. The results are obtained by Gibbs sampler algorithm with 5000 repetitions and are given in Table 5.

Table 5 illustrates that, for $n = 100$ and $m = 50$, the rejection rates of sub-hypotheses H_{01}, H_{02} and H_{03} are more than 60% at 5% level when the parameter exhibits a change, while it is only 6.6% when the parameter is stable (without change). For example, for the set of parameters $\mu_1 = 0.0$, $\mu_2 = 0.5$, $\phi_1 = 0.3$, $\phi_2 = 0.3$ and $\sigma_1^2 = 1.0$, $\sigma_2^2 = 0.5$, the rejection rate of the sub hypothesis H_{01} is 0.630, for H_{02} is 0.004 and for H_{03} is 0.711. We note that the parameter ϕ is stable. For the last set of parameters, $\mu_1 = 0.0$, $\mu_2 = 0.0$, $\phi_1 = 0.3$, $\phi_2 = 0.3$ and $\sigma_1^2 = 0.5$, $\sigma_2^2 = 0.5$, the three parameters are assumed to be stable, the rejection rate of sub hypotheses H_{01}, H_{02} and H_{03} are respectively 0.006, 0.009 and 0.066. Therefore, the test detects well the autoregressive parameters that are subjects to a change.

It can be seen that the rejection rates of sub-hypotheses H_{01}, H_{02} and H_{03} of AR(1) model increases with the sample size. For $n = 200$, $m = 100$, $\mu_1 = 0.0$, $\mu_2 = 0.5$, $\phi_1 = 0.3$, $\phi_2 = 0.3$ and $\sigma_1^2 = 1.0$, $\sigma_2^2 = 0.5$, the rejection rate of the sub-hypotheses H_{01}, H_{02} and H_{03} of AR(1) are respectively 0.810, 0.018 and 0.877. However, they are respectively only 0.220, 0.007 and 0.516 for $n = 50$ and $m = 25$. Therefore, the sample size has a positive impact on the Bayesian significance test of change in the parameters of autoregressive time series models.

5. Application

In this section, we illustrate our test procedures using three data sets, which are the monthly average soybean, corn and wheat prices achieved by farmers in Illinois from one January 1960 to one December 1984. The prices are given in dollars per bushel. The price y_t is observed each month from one January 1960 until one December 1984 with sample of 300 observations. Data used in this analysis cab be found in (https://farmdoc.illinois.edu/decision-tools/illinois-average-farm-price-received-database). The sample size is 300. The series are plotted in Figures 6 (a)-(c), Berkes et al. (2011) study two real data sets. The first sample consists of monthly average corn prices and the second sample consists of monthly average soybean prices achieved by farmers in Illinois from January 1960 to November 2008. The results of their statistical test indicate that the changes from an AR(1) to a threshold AR(1) occurred around July 1971 (Corn) and October 1974 (Soybeans).



(a) (b) (c)

Figure 6. Monthly average for corn prices (a); Soybean prices (b) and wheat prices (c) from 1960 to 1984.

We are interested of wither there is any evidence for the existence of a change in the parameter of AR(1) model. A visual inspection of this series in Figures 6(a)-(c) seem to suggest that there might be a change in the parameters of the series. By application of the Gibbs sampler algorithm with 10,000 repetitions we approximate the unconditional p-values for the hypothesis $H_{01} : \delta = 0$, $H_{012} : \rho = 0$ and $H_{03} : \tau = 1$ and the posterior estimates of the change point m. The results are given in Tables 6.

Table 4. The unconditional p-value of H_{01}, H_{02} and $H_{0"}$ with different values of $\delta = \mu_2 - \mu_1$, $\rho = \phi_2 - \phi_1$ and $\tau = \sigma_2^2/\sigma_1^2$.

Parameters / p-values	$\mu_1 = 0.0$, $\mu_2 = 0.5$ $\phi_1 = 0.3$, $\phi_2 = -0.2$ $\sigma_1^2 = 0.5$, $\sigma_2^2 = 0.5$	$\mu_1 = 0.0$, $\mu_2 = 0.5$ $\phi_1 = 0.3$, $\phi_2 = 0.3$ $\sigma_1^2 = 1.0$, $\sigma_2^2 = 0.5$	$\mu_1 = 0.0$, $\mu_2 = 0.0$ $\phi_1 = 0.3$, $\phi_2 = -0.2$ $\sigma_1^2 = 1.0$, $\sigma_2^2 = 0.5$	$\mu_1 = 0.0$, $\mu_2 = 0.0$ $\phi_1 = 0.3$, $\phi_2 = 0.3$ $\sigma_1^2 = 0.5$, $\sigma_2^2 = 0.5$
$P_{\delta=0\|y}$	$4.38.10^{-6}$	0.0187	0.3804	0.3919
$P_{\rho=0\|y}$	0.0029	0.4134	0.0551	0.3728
$P_{\tau=1\|y}$	0.75086	0.0146	0.0211	0.5328

Table 5. Rejection rate of subnull H_{01}, H_{02} and $H_{0"}$ at 5% level for 1000 samples with different values of n, m, $\delta = \mu_2 - \mu_1$, $\rho = \phi_2 - \phi_1$ and $\tau = \sigma_2^2/\sigma_1^2$.

	Parameters / Subnull	$\mu_1 = 0.0$, $\mu_2 = 0.5$ $\phi_1 = 0.3$, $\phi_2 = -0.2$ $\sigma_1^2 = 0.5$, $\sigma_2^2 = 0.5$	$\mu_1 = 0.0$, $\mu_2 = 0.5$ $\phi_1 = 0.3$, $\phi_2 = 0.3$ $\sigma_1^2 = 1.0$, $\sigma_2^2 = 0.5$	$\mu_1 = 0.0$, $\mu_2 = 0.0$ $\phi_1 = 0.3$, $\phi_2 = -0.2$ $\sigma_1^2 = 1.0$, $\sigma_2^2 = 0.5$	$\mu_1 = 0.0$, $\mu_2 = 0.5$ $\phi_1 = 0.3$, $\phi_2 = -0.2$ $\sigma_1^2 = 1.0$, $\sigma_2^2 = 0.5$	$\mu_1 = 0.0$, $\mu_2 = 0.0$ $\phi_1 = 0.3$, $\phi_2 = 0.3$ $\sigma_1^2 = 0.5$, $\sigma_2^2 = 0.5$
$n = 50$	$H_{01} : \delta = 0$	0.765	0.220	0.000	0.730	0.003
$m = 25$	$H_{02} : \rho = 0$	0.386	0.007	0.393	0.393	0.001
	$H_{03} : \tau = 1$	0.095	0.443	0.516	0.537	0.052
$n = 100$	$H_{01} : \delta = 0$	0.997	0.630	0.000	0.997	0.006
$m = 50$	$H_{02} : \rho = 0$	0.797	0.004	0.770	0.787	0.009
	$H_{03} : \tau = 1$	0.159	0.711	0.825	0.846	0.066
$n = 200$	$H_{01} : \delta = 0$	1	0.810	0.003	0.998	0.009
$n = 100$	$H_{02} : \rho = 0$	0.987	0.018	0.810	0.921	0.000
	$H_{03} : \tau = 1$	0.210	0.865	0.877	0.944	0.016

Table 6. Unconditional p-values of the hypothesis H_{01}, H_{02} and H_{03} and posterior mode and median of change point m for monthly average for soybeans, for corn and for wheat prices.

Dataset	mode	median	$P_{\delta=0\|y}$	$P_{\rho=0\|y}$	$P_{\tau=1\|y}$
Soybean	154	155	0.2001	0.0866	0.0220
Corn	155	155	0.3641	0.1066	0.0390
Wheat	163	163	0.2832	0.2809	0.0014

Table 6 shows some numerical results of the series of monthly average soybeans, corn and wheat prices from one January 1960 to one December 1984. Posterior mode of the change point m indicates that the changes occurred at time $m = 154$, that corresponds to around October 1972 for Soybeans, at time $m = 155$, that corresponds to November 1972 for Corn and at time $m = 163$, that corresponds to around July 1973 for Wheat. As, the smaller the p-value, more the strength of the evidence against H_0 is significant, the values of the p-values indicate that there is evidence against the equality of the variances of the three series of observations. Thus, the unconditional p-values $P_{\delta=0|y}$, $P_{\rho=0|y}$ and $P_{\tau=1|y}$ of the hypotheses H_{01}, H_{02} and H_{03}, respectively, indicate that the no change in the variance of the series of Saybeans, Corn and Wheat is rejected at 5% significance level. While, the no change in the mean cannot be rejected even at 20% significance level for the three crops. For the change in the autocorrelation coefficient it can be rejected at 10% significance level for Saybeans and it can hardly be rejected at 10% significance level for Corn, and cannot be rejected even at 20% significance level for Wheat. Consequently, the results in Table 6 indicate that the prices of Soybeans, Corn and wheat have undergone a significant variation in the variance parameter since October 1972 for Soybeans, since November 1972 for Corn and since July 1973 for Wheat. Period which corresponds to the beginning of the world food crisis of the 1970s (FAO (2009)), a time from mid-1972 to mid-1975 (Gerlach (2015)).

6. Conclusions, limitations, and future research

In this paper, we have investigated a Bayesian detection of change in the parameters of an autoregressive process of known order p. The model is subjected to a change in $p + 2$ parameters, the mean, the variance of the error terms and the p autoregressive parameters at an unknown time point. We derived the conditional posterior distributions of the change point, of the magnitude of the shift in the mean, of the magnitude of the shift in the autocorrelation coefficients and of the variance ratio. An unconditional Bayesian significance test of change based on the calculation of the p-values is determined. The test detects separately the autoregressive parameters which are subject to a change at an unknown time m. The Gibbs sampler algorithm is employed to estimate the model parameters. The performance of the test has been investigated on simulated and real data sets. We showed how inferences can be made readily by using the Bayesian significance test based on the highest posterior density credible sets for detecting a change of an individual parameter of autoregressive models. Also, we have showed the impact of the sample size on the Bayesian significance test of change. We have illustrated the application of the methods using three real datasets available in the literature. The datasets are the monthly average soybean, corn and wheat prices achieved by farmers in Illinois from one January 1960 to one December 1984. Results obtained report the existence of a change point in all three datasets. The change points obtained correspond exactly to the beginning of the food crisis which occurred in the early 1970s. A possible limitation of the adopted approach might be associated with the estimation of all the parameters of the model, a similar approach could be adopted to estimate all the parameters to performance residual analysis. Moreover, it would be interesting to extend the study to examine the problem of multiple structural change points and to study the case where the order of the autoregressive model is unknown.

Author contributions Conceptualization; data curation; formal analysis; investigation; methodology; software; supervision.validation; visualization; writing-original draft preparation; and writing-review and editing: A.S. The author (A.S.) has read and agreed the published version of the paper.

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