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Contents

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Functional and Nonparametric Statistics
Research Paper

Nonparametric relative error regression for functional time series data under random censorship

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Abstract

In this paper, we investigate the asymptotic properties of a nonparametric estimator of the relative error regression given a dependent functional explanatory variable, in the case of a scalar censored response. We use the mean squared relative error as a loss function to construct a nonparametric estimator of the regression operator of these functional censored data. We establish the almost surely convergence (with rates) and the asymptotic normality of the proposed estimator. A simulation study and real data application are performed to lend further support to our theoretical results and to compare the quality of predictive performances of the relative error regression estimator than those obtained with standard kernel regression estimates.

Keywords: Almost surely convergence · α-mixing data · Censored data · Functional data analysis · Mean square relative error · Nonparametric estimation.

Mathematics Subject Classification: Primary 62G35 · Secondary 62G20.

1. Introduction

Functional data analysis is a branch of statistics that has gained popularity in recent years, either mathematically or in terms of applications. There are numerous practical applications for this data format, such as continuous phenomena (climatology, economics, linguistics, medicine, and so on). Since the publication of Ramsay and Dalzell (1991)'s work, numerous developments have been examined in order to produce theories and methodologies that are based on functional data (Almanjahie et al., 2020).

The monographs of Ramsay and Silverman (2005) provide an overview of both the theoretical and practical elements of functional data analysis, whereas the monographs of Ferraty and Vieu (2006) provide an overview of nonparametric techniques. Numerous nonparametric models have been developed. For example, Ferraty and Vieu (2004) established the strong consistency of the regression function when the explanatory variable is functional and the response is scalar, and their study extended to non-standard regression problems such as time series prediction and curve discrimination (Ferratyet al., 2002; Ferraty and Vieu, 2003); for robust estimation, see also Attouch et al. (2009).

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Masry (2005) establishes the asymptotic normality of the same estimator under an α-mixing assumption. According to Dabo-Niang (2004), density estimation in a Banach space was investigated, as well as the density estimation of a diffusion process with respect to the Wiener measure. Ferraty and Vieu (2006) introduced the kernel type estimation (Azevedo et al., 2011) of some characteristics of the conditional cumulative distribution function (CDF) as well as the successive derivatives of the conditional density; the almost complete convergence (ACC) with rates for the kernel type estimates is established and illustrated by an application to El Niño data. It is common practice to estimate the regression function by minimizing the mean-squared loss function. When data contains outliers, this loss function is predicated on some restrictive constraints, such as the variance of the residual being equal for all observations. As a result, in order to overcome this limitation, we investigate an alternate strategy that allows us to create an effective predictor even when the data is influenced by the existence of outliers. As a result, the constraints of classical regression are addressed in this study by estimating the regression function with respect to the mean squared relative error (MSRE). The latter is a better indicator of a predictor's performance than the usual inaccuracy in the prediction.

The literature on the relative error regression in nonparametric functional data analysis (NFDA) is still limited. The first consistent results were obtained in by Campbell and Donner (1989), where relative regression was used as a classification tool. Jones et al. (2008) studied the nonparametric prediction via relative error regression. They investigated the asymptotic properties of an estimator minimizing the sum of the squared relative errors by considering both (kernel method and local linear approach). Recently, Mechab and Laksaci (2016) analyzed this regression model when the observations are weakly dependent. For spatial data, Attouch et al. (2017) proved the almost complete consistency and the asymptotic normality of this estimator. Fetitah et al. (2020) investigated the relative error in functional regression under random censorship when data are independent.

Nonparametric analysis of incomplete functional data, on the other hand, has a limited extensive literature. There are limited works on this issue (for example, Altandji et al. (2018) estimates the relative error in functional regression under the random left-truncation model). Carbonez et al. (1995) presented the kernel estimator of classical regression in the right censorship model, and improved it in Ould-Saïd and Guessoum (2008). To estimate the conditional quantile when regressors are functional, this approach was later employed by Horrigue and Ould-Saïd (2014). Additionally, using truncated data, Helal and Ould-Saïd (2016) used the same model.

In this paper we define and study a new estimator of the regression function when the interest random variable is subject to random right-censoring and the explanatory variable is functional. Notice that the main feature of our approach is to develop a prediction model alternative to the classical regression which is not sensitive to the presence of the outliers.

The paper is organized as follows. In Section 2 we define our parameter of interest and its corresponding estimators. In Section 3 we give some assumptions and state an almost sure (AS) consistency and asymptotic normality for the proposed estimator. A simulation study and real data application are performed in Section 4, whereas the technical details and the proofs are deferred to Section 4.2.

2. Model

2.1 Background

Let consider that (Y_i, X_i), for $i = 1 \dots n$, is a stationary $\alpha-$mixing couples, where Y_i is real-valued and X_i takes values in some functional space $\mathcal{F}$. Assume that $\mathbb{E}|Y_i| < \infty$ and define the regression functional as

$$r(x) = \mathbb{E}[Y_i | X_i = x], \quad x \in \mathcal{F}, \quad \forall i \in \mathbb{N}. \tag{1}$$

The model given in Equation (1) can be written as

$$Y_i = r(X_i) + \epsilon_i, \quad i = 1, \ldots, n,$$

where ϵ_i is a random variable such that $\mathbb{E}[\epsilon_i | X_i] = 0$ and $\mathbb{E}[\epsilon_i^2 | X_i] = \sigma^2(X_i) < +\infty$.

Unlike to the multivariate case, there exists various versions of the functional regression estimate. However, all these versions are based on two common procedures. The first one is the functional operator which is supposed smooth enough to be locally well approximated by a polynomial. The second one is the use of the least square error given by

$$r(x) = \arg\min_{r^*} \left(\mathbb{E}\left[(Y - r^*(x))^2 | X = x \right] \right), \tag{2}$$

as a loss function to determine the estimates of r. In complete data, a typical kernel regression estimator based on Equation (2) (Ferraty et al., 2007) is given by

$$r_n(x) = \frac{\sum_{i=1}^n Y_i K\left(h^{-1} d(x, X_i)\right)}{\sum_{i=1}^n K\left(h^{-1} d(x, X_i)\right)},$$

where K is a kernel and $(h := h_n)$ is a sequence of bandwidths.

For results on both theoretical and application points of view considering independent or dependent case, we refer the reader to the studies of Attouch et al. (2017) and Chahad et al. (2017). Note that Amiri et al. (2014) analyzed the regression function of a real random variable with functional explanatory variable by using a recursive nonparametric kernel approach.

In the presence of right random censoring, the problem has been analyzed by Buckley and James (1979) using parametric methods. For nonparametric approaches, we refer to Amiri and Khardani (2018) and Stute (1993). Some asymptotic properties were established with a particular application to the conditional mode and quantile by Chaouch and Khardani (2015) and Khardani and Thiam (2016). Horrigue and Ould-Saïd (2014) considered a regression quantile estimation for dependent functional data. Nevertheless, the use of previous loss function given in Equation (2) as a measure of prediction performance may be not suitable in some situation. In particular, the presence of outliers can lead to unreasonable results since all variables have the same weight. Now, to overcome this limitation we propose to estimate the function r by an alternative loss function.

In the relative regression analysis, r is obtained by minimizing the MSRE, that is, $r(x)$ is the solution of the optimization problem:

$$r(x) = \arg\min_{r^*} \left(\mathbb{E}\left[\left(\frac{Y - r^*(X)}{Y} \right)^2 | X = x \right] \right).$$

As mentioned in Jones et al. (2008), where outlier data are present and the response variable of the model is positive, the MSRE is minimized.

It is clear that this criterion is a more meaningful measure of prediction performance than the least squares error, in particular when $Y > 0$, it often is that the ratio of prediction error to the response level, $(Y - r(X))/Y$, is of prime interest: the expected squared relative loss, $\mathbb{E}[\{(Y - r(X))/Y\}^2 | X]$, which is the MSRE, is minimized (specially in the presence of outliers). Moreover, the solution of this problem can be expressed by the ratio of first two conditional inverse moments of Y given X. As discussed by Park and Stefanski (1998), for $Y > 0$

$$r(x) = \frac{\mathbb{E}\left[Y^{-1}|X = x\right]}{\mathbb{E}\left[Y^{-2}|X = x\right]} := \frac{g_1(x)}{g_2(x)}, \tag{3}$$

where $g_l(x) = \mathbb{E}[Y^{-l}|X = x]$, for $l = 1, 2$, with r being the best MSRE predictor of Y given $X = x$.

2.2 Construction of the estimator

To construct our estimator, let us recall that in the case of complete data, a well-known estimator of the regression function is based on the Nadaraya-Watson weights. Let $\{Z_i = (X_i, Y_i)_{1\leq i\leq n}\}$ be n pairs, identically distributed as $Z = (X, Y)$ and valued in $\mathcal{F} \times \mathbb{R}$, where $(\mathcal{F}, d)$ is a semi-metric space (that is, X is a functional random variable (FRV) and d a semi-metric). Let x be a fixed element of $\mathcal{F}$. For the complete data, see Demongeot et al. (2016).

It is well known that the kernel estimator of Equation (3) is given by

$$\hat{r}(x) = \frac{\sum_{i=1}^{n} Y_i^{-1} K\left(\frac{d(x, X_i)}{h}\right)}{\sum_{i=1}^{n} Y_i^{-2} K\left(\frac{d(x, X_i)}{h}\right)} := \frac{\hat{g}_1(x)}{\hat{g}_2(x)},$$

where $\hat{g}_l(x) = \sum_{i=1}^{n} Y_i^{-l} K(d(x, X_i)/h)/(n\mathbb{E}(K(d(x, X_1)/h)))$, for $l = 1, 2$, with K is an asymmetrical kernel and $h = h_n$ (depending on n) is a strictly positive real. It is a functional extension of the familiar Nadaraya-Watson estimate. The main change comes from the semi-metric d which measures the proximity between functional objects.

In the censoring case, instead of observing the lifetimes Y, which has a continuous CDF F, we observe the censored lifetimes of items under study, that is, assuming that $(C_i)_{1\leq i\leq n}$ is a sequence of independent and identically distributed censoring random variable (RV) with common unknown continuous CDF G. Then, in the right censorship model, we only observe the n pairs (T_i, δ_i) with $T_i = Y_i \wedge C_i$ and $\delta_i = \mathbb{1}_{\{Y_i \leq C_i\}}$, for $1 \leq i \leq n$, where $\mathbb{1}_A$ denotes the indicator function of the set A.

In what follows, we define the endpoints of F and G by $\tau_F = \sup\{t\colon \bar{F}(t) > 0\}$, and $\tau_G = \sup\{t\colon \bar{G}(t) > 0\}$ where $\bar{F}(x) = 1 - F(x)$ and $\bar{G}(x) = 1 - G(x)$. We assume that $\tau_F < \infty$ and $\bar{G}(\tau_F) > 0$, (this implies $\tau_F < \tau_G$).

In censorship model, only the $(X_i, T_i, \delta_i)_{1\leq i\leq n}$ are observed. We define $\widetilde{r}(x)$ as an estimate of $r(x)$ by

$$\widetilde{r}(x) = \frac{\sum_{i=1}^{n} \frac{\delta_i T_i^{-1}}{\bar{G}(T_i)} K\left(\frac{d(x, X_i)}{h}\right)}{\sum_{i=1}^{n} \frac{\delta_i T_i^{-2}}{\bar{G}(T_i)} K\left(\frac{d(x, X_i)}{h}\right)} =: \frac{\widetilde{g}_1(x)}{\widetilde{g}_2(x)}, \tag{4}$$

where

$$\widetilde{g}_l(x) = \frac{\sum_{i=1}^{n} \frac{\delta_i T_i^{-l}}{\bar{G}(T_i)} K\left(\frac{d(x,X_i)}{h}\right)}{n\mathbb{E}\left(K\left(\frac{d(x,X_1)}{h}\right)\right)}, \quad l = 1, 2.$$

In practice, G is unknown. We use the Kaplan-Meier estimator (Deheuvels and Einmahl, 2000) of $\bar{G}$ given by

$$\bar{G}_n(t) = \begin{cases} \prod_{i=1}^{n} \left(1 - \frac{1-\delta_{(i)}}{n-i+1}\right)^{\mathbb{1}_{\{T_{(i)} \leq t\}}}, \text{ if } t \leq T_{(n)}, \\ 0, \text{ otherwise,} \end{cases}$$

where $T_{(1)} \leq \cdots \leq T_{(n)}$ are the order statistics of $(T_i)_{1 \leq i \leq n}$ and $\delta_{(i)}$ is the concomitant of $T_{(i)}$. Therefore, the estimator of r (Fetitah et al., 2020) is stated as

$$\widetilde{r}_n(x) = \frac{\sum_{i=1}^{n} \frac{\delta_i T_i^{-1}}{\bar{G}_n(T_i)} K\left(\frac{d(x,X_i)}{h}\right)}{\sum_{i=1}^{n} \frac{\delta_i T_i^{-2}}{\bar{G}_n(T_i)} K\left(\frac{d(x,X_i)}{h}\right)} := \frac{\widetilde{g}_{1,n}(x)}{\widetilde{g}_{2,n}(x)}, \tag{5}$$

where

$$\widetilde{g}_{l,n}(x) = \frac{\sum_{i=1}^{n} \frac{\delta_i T_i^{-l}}{\bar{G}_n(T_i)} K\left(\frac{d(x,X_i)}{h}\right)}{n\mathbb{E}\left(K\left(\frac{d(x,X_1)}{h}\right)\right)}, \quad l = 1, 2.$$

Remark 2.1 In Equations (4) and (5), the sums are taken for the subscripts i, where $\bar{G}_n(T_i) \neq 0$ and $\bar{G}(T_i) \neq 0$. The same convention is followed in the forthcoming formulas. Note that, under the assumptions on the model, the sets $\{i, \bar{G}(T_i) = 0\}$ and $\{i, \bar{G}_n(T_i) = 0\}$ are $\mathbb{P}$-negligible.

3. Assumptions and main results

3.1 General context

Throughout this paper, x is a fixed element of the functional space $\mathcal{F}$. To formulate our assumptions, some notations are required. and we denote by $\mathcal{N}_x$ a neighborhood of the point x. Hereafter, when no confusion is possible, we denote by c and c' some strictly positive generic constants.

Let $B(x, h)$ be the closed ball centered at x with radius h, and consider the CDF of $d(x, X)$ defined by

$$\varphi_x(h) = \mathbb{P}(X \in B(x, h)) = \mathbb{P}(d(x, X) \leq h),$$

with h being positive and satisfies $\varphi_x(0) = 0$ and $\varphi_x(h) \to 0$ when $h \to 0$. Let us consider the following definition.

Definition 3.1 Let $(Z_n)_{n\in\mathbb{N}}$ be a sequence of RVs. Given a positive integer n, set

$$\alpha(n) = \sup_k \sup \left\{ |\mathbb{P}(A\cap B) - \mathbb{P}(A)\mathbb{P}(B)|, A \in \mathcal{F}_1^k(Z) \text{ and } B \in \mathcal{F}_{k+n}^{\infty}(Z) \right\},$$

where $\mathcal{F}_i^k(Z)$ denotes the $\sigma-$field generated by $\{Z_j, i \le j \le k\}$. The sequence is said to be $\alpha-$mixing if the mixing coefficient $\alpha(n) \to 0$ when $n \to \infty$.

3.2 Asymptotic consistency

Our main first result is the pointwise almost sure convergence. In order to state this result, we need some assumptions which are gathered together in order to make our results reading easier. In what follows, we assume that the following assumptions hold:

(H1) $\mathbb{P}(X \in B(x,h)) =: \varphi_x(h) > 0$, for all $h > 0$.
(H2) For all $(x_1, x_2) \in \mathcal{N}_x^2$, we have

$$|g_l(x_1) - g_l(x_2)| \le c d^{k_l}(x_1, x_2) \text{ for an integer } k_l > 0 \text{ and } l = 1, 2.$$

(H3) The kernel K is a bounded and Lipschitzian function on its support $(0, 1)$ and satisfying:

$$0 < c \le K(x) \le c' < +\infty.$$

(H4) The bandwidth h satisfies $h \to 0$, $\log(n)/(n\varphi_x(h)) \to 0$ as $n \to \infty$.
(H5) The inverse moments of the response variable verify:

$$\text{for all } m \ge 2, \quad \mathbb{E}[Y^{-m}|X = x] < c_m < \infty.$$

where c_m is positive constant.
(H6)

(i) $(X_n, Y_n)_{n\ge1}$ is a sequence of stationary $\alpha-$mixing RVs with coefficient $\alpha(n) = O(n^{-a})$, for some $a \in (0, \infty)$.
(ii) $(C_n)_{n\ge1}$ and $(X_n, Y_n)_{n\ge1}$ are independent.

(H7) For all $i \ne j$, $\mathbb{E}[Y_i^{-1}Y_j^{-2}|(X_i, X_j)] \le c < \infty$, and

$$0 < \sup_{i\ne j} \Big\{ \mathbb{P}\left((X_i, X_j) \in B(x,h) \times B(x,h)\right) \Big\} = O\left(\frac{(\varphi_x(h))^{(a+1)/a}}{n^{1/a}}\right).$$

(H8) There exists $\eta > 0$, such that, $cn^{\frac{3-a}{a+1}+\eta} \le \varphi_x(h) \le c' n^{\frac{1}{1-a}}$, with $a > 2$.

We are in state to give our main result.

Theorem 3.2 Under Assumptions (H1)-(H8), we have

$$|\widetilde{r}_n(x) - r(x)| = O(h^{k_1}) + O(h^{k_2}) + O_{\mathrm{AS}}\left(\sqrt{\frac{\log(n)}{n\varphi_x(h)}}\right).$$

3.3 Asymptotic normality

Here, we study of the asymptotic normality of $\widetilde{r}_n(x)$. To do that, we replace assumptions (H1), (H3) and (H4) respectively by the following hypotheses:

(N1) The concentration property (H1) holds. Moreover, there exists a function $\chi_x(.)$ such that,

$$\text{for all} \quad s \in [0,1], \quad \lim_{r \to 0} \frac{\varphi_x(sr)}{\varphi_x(r)} = \chi_x(s).$$

(N2) For $\gamma \in \{1,2\}$, the functions $\Psi_\gamma(x) = \mathbb{E}\left[g_\gamma(X) - g_\gamma(x) | d(x,X) = x\right]$ are derivable at zero.
(N3) The kernel function K satisfies (H3) and is a differentiable function on $]0,1[$ where its first derivative function K' is such that: $-\infty < c < K'(x) < c' < 0$.
(N4) The small ball probability satisfies: $n\varphi_x(h) \longrightarrow \infty$.
(N5) For $m \in \{1,2,3,4\}$, the functions $q_m(x) = \mathbb{E}[\bar{G}(Y)^{-1}Y^{-m}|X = x]$ are continuous in a neighborhood of x.

Assumption (H1) is the same as that used by Ferraty and Vieu (2006) which is linked to the functional structure of the functional covariate. Assumptions (H2), (H3) and (H7) deal with the functional aspect of the covariate and the associated small ball probability techniques used in this paper. Assumptions (H6) and (H8) specify the model and the rate of mixing coefficient. Condition (N5) stands as regularity condition that is useful to establish the asymptotic properties of the estimators. Assumptions (H3), (H4), (N3) and (N4) concern the kernel K and the smoothing parameter h and are technical conditions.

The fractal or geometric process is a family of infinite dimensional processes for which the small balls have the property $\varphi_x(t) = \mathbb{P}\left(\|x - X\| < t\right) \sim c_x t^\gamma$, where c_x and γ are positive constants. In this case, setting $h_n = An^{-a}$ with $0 < a < 1$ and $0 < A$ implies $\varphi_x(h) = c_x A n^{-\gamma a}$. Thus, (H1), (H4) and (H8) hold when $\gamma < 1/a$.

Theorem 3.3 Under Assumptions (H6)-(H8)and (N1)-(N5), we have

$$\left(\frac{n\varphi_x(h)}{\sigma^2(x)}\right)^{1/2} \left(\widetilde{r}_n(x) - r(x) - hB_n(x) - o(h)\right) \xrightarrow{\mathcal{D}} \mathcal{N}(0,1), \quad \text{as} \quad n \to \infty.$$

where $\xrightarrow{\mathcal{D}}$ denotes convergence in distribution,

$$B_n(x) = \frac{(\Psi_1'(0) - r(x)\Psi_2'(0))\beta_0}{\beta_1 g_2(x)}$$

and

$$\sigma^2(x) = \frac{(q_2(x) - 2r(x)q_3(x) + r^2(x)q_4(x))\beta_2}{\beta_1^2} \neq 0,$$

with $\beta_0 = K(1) - \int_0^1 (sK(s))'\chi_x(s)\mathrm{d}s$ and $\beta_j = K^j(1) - \int_0^1 (K^j)'(s)\chi_x(s)\mathrm{d}s \neq 0$, for $j = 1,2$.

Remark 3.4 (Comeback to complete data). In absence of censoring ($\bar{G}(x) = 1$), the asymptotic variance becomes

$$\sigma^2(x) = \frac{(a_2(x) - 2r(x)a_3(x) + r^2(x)a_4(x))\beta_2}{\beta_1^2},$$

where $a_j(x) = \mathbb{E}[Y^{-j}|X = x]$, which is the result obtained by Demongeot et al. (2016).

4. Simulation and application

4.1 Simulation study

In this section, we treat a simulation example to show the behaviour of our estimator $\widetilde{r}_n(x)$ and to compare the sensitivity to outliers of the classical regression defined as the conditional expectation $m(x) = \mathbb{E}[Y|X = x]$ estimated by

$$\widehat{m}(x) = \frac{\sum_{i=1}^{n} \frac{\delta_i T_i}{\bar{G}_n(T_i)} K\left(h^{-1} d\left(x, X_i\right)\right)}{\sum_{i=1}^{n} K\left(h^{-1} d\left(x, X_i\right)\right)},$$

and the relative error estimator $\widetilde{r}_n(x)$ previously defined. To do this, we consider the classical nonparametric functional regression model stated as

$$Y = r(X) + \epsilon,$$

where the operator r is defined by $r(X) = 10/[1 + \int_0^1 X^2(t)\mathrm{d}t]$.
We consider two diffusion processes on the interval $[0, 1]$, $Z_1(t) = 2 - \cos(\pi t W)$ and $Z_2(t) = \cos(\pi t W)$, and we take $X(t) = AZ_1(t) + (1 - A)Z_2(t)$, where A is a Bernoulli distributed RV and W is an α-mixing process generated by the model expressed as

$$W_i = \frac{1}{\sqrt{2}}\left(W_{i-1} + \eta_i\right), \quad i = 1, \ldots, 200,$$

with η_i being centered Gaussian distributed RVs with variance 0.5 and independent of η_i. We carried out the simulation with $n = 200$ sample of the curve X. The error variable $\epsilon_i \sim \mathcal{N}(0, 0.5)$. We also, simulate n independent and identically distributed RV C_i, for $i = 1, \ldots, n$, with law $\mathcal{E}(\lambda)$ (that is, exponentially distributed with density $\lambda e^{-\lambda x}\mathbb{1}_{\{x\geq 0\}}$). Simulated data from our model are plotted in Figure 1. To compute our estimator based on the observed data (X_i, T_i, δ_i), for $i = 1, \ldots, n$, where $T_i = Y_i \wedge C_i$ and $\delta_i = \mathbb{1}_{\{Y_i \leq C_i\}}$.

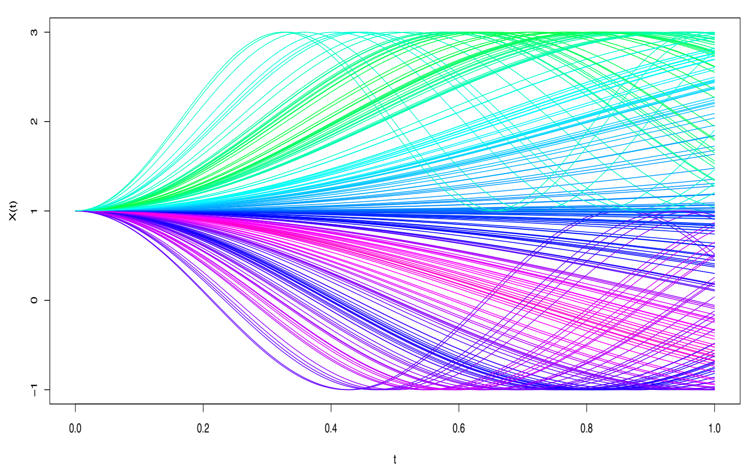

Figure 1. The curves $X_{i=1,\ldots,100}(t)$, for $t \in [0, 1[$.

We choose the quadratic kernel defined by

$$K(x) = \frac{3}{2}\left(1 - x^2\right)\mathbb{1}_{(0,1)}.$$

In practice, the semi-metric choice is based on the regularity of the curves X which are under analysis. In our case, we take the semi-metric based on the second derivatives of the curves X. More precisely, we take

$$d(X_i, X_j) = \left(\int_0^1 (X_i''(t) - X_j''(t))^2 \mathrm{d}t\right)^{1/2} \quad \forall X_i, X_j \in \mathcal{F},$$

where X'' denotes the second derivative of the curve X. For the bandwidth, we choose the automatic selection with a cross validation procedure introduced by (Ferraty and Vieu, 2006, Ch.13).

We split the data generated from the model above into two subsets: a training sample (X_i, T_i, δ_i), for $i = 1, \ldots, 150$, and a test sample (X_j, T_j, δ_j), for $j = 151, \ldots, 200$. Then, we calculate the estimator $\tilde{r}(X_j)$ for any $j \in \{151, \ldots, 200\}$.

The performance of both estimators is described by the mean squared error (MSE) formulated as

$$\mathrm{MSE} = \frac{1}{50}\sum_{j=151}^{200} (r(X_j) - \tilde{r}(X_j))^2,$$

where $\tilde{r}(X_j)$ means the estimator of both regression models and $r(X_j)$ the response variable. We note that the result of our simulation study is evaluated over 100 independent replications.

The obtained results are shown in Figure 2 with the censorship rate CR $= 20.67\%$. It is clear that there is no meaningful difference between the two estimation methods: the classical kernel estimator (CKE) has an $\mathrm{MSE}_{\mathrm{CKE}} = 0.2209$, whereas the relative error estimator (REE) has an $\mathrm{MSE}_{\mathrm{REE}} = 0.1579$.

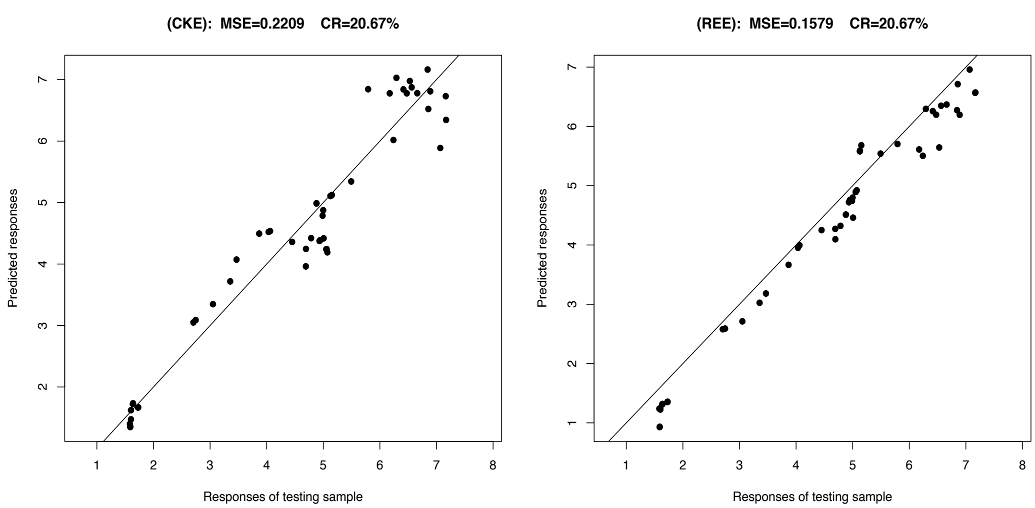

Figure 2. comparison between the CKE and the REE without outliers.

The results of a second illustration are given in Table 1, where from we observe that, in the presence of outliers $(0, 10, 20)$ with different values of CR $= 3\%, 30\%, 60\%)$, the relative error regression performs better than the classical method, even if the MSE of both methods increases substantially relatively to the number of the perturbed points and censorship rate, it remains very low in terms of the relative error.

Table 1. MSE of the CKE and REE according to numbers of introduced artificial outliers and different censorship rate.

Number of artificial outliers	→ CR ↓	0	10	20
	3%	0.0921	2856.646	6499.6945
Classical kernel estimator	30%	0.8766	14126.2706	19358.5386
MSE_{CKE}	60%	2.8038	32182.8188	56681.7038
	3%	0.0551	0.0579	0.0665
Relative error estimator	30%	0.0949	0.1048	0.1258
MSE_{REE}	60%	0.1455	0.1903	0.2712

Our main application of Theorem 3.3 is to build confidence intervals (CIs) for the true value of $r(x)$ given curve $X = x$. A plug-in estimate for the asymptotic standard deviation $(n\varphi_x(h)/\sigma^2(x))^{1/2}$ and the bias term $hB_n(x) + o(h)$. Precisely, we estimate $q_m(x)$ by means of

$$\widetilde{q}_m(x) = \frac{\sum_{i=1}^n K_i \delta_i \bar{G}_n^{-2}(T_i) T_i^{-m}}{\sum_{i=1}^n K_i},$$

whereas we estimate empirically β_1 and β_2 by using

$$\hat{\beta}_1 = \frac{1}{n\varphi_x(h)} \sum_{i=1}^n K_i \quad \text{and} \quad \hat{\beta}_2 = \frac{1}{n\varphi_x(h)} \sum_{i=1}^n K_i^2.$$

Thus, the practical estimator of the normalized deviation is stated as

$$\widetilde{\sigma}_n(x) = \left(\frac{\left(\sum_{i=1}^n K_i^2\right) \left(\widetilde{q}_2(x) - 2\widetilde{r}(x)\widetilde{q}_3(x) + \widetilde{r}^2(x)\widetilde{q}_4(x)\right)}{\left(\sum_{i=1}^n K_i\right)^2 \widetilde{q}_2^2(x)} \right)^{1/2}.$$

We point out that the function φ_x do not intervene in the calculation of the CI by simplification. Hence, the approximate $(1 - \zeta/2) \times 100\%$ CI for $r(x)$, for any $x \in \mathcal{F}$, is given by

$$\left[\widetilde{r}_n(x) - z_{1-\zeta/2}\widetilde{\sigma}_n(x), \widetilde{r}_n(x) + z_{1-\zeta/2}\widetilde{\sigma}_n(x)\right],$$

where $z_{1-\zeta/2}$ denotes the $(1 - \zeta/2) \times 100$th quantile of the standard normal distribution.

In order to compare our CI with that of the classical regression (Ferraty et al., 2007), we have

$$\sqrt{n\varphi_x(h)} \frac{\beta_1}{\sigma_\epsilon(x)\sqrt{\beta_2}} \left(\widehat{m}(x) - m(x)\right) \xrightarrow{\mathcal{D}} \mathcal{N}(0, 1),$$

where $\sigma_\epsilon^2(x) = \mathbb{E}\left[(Y - m(x))^2 | X = x\right]$ and β_1, β_2 are define previously.

With simple calculus, we can estimate $\sigma_\epsilon^2(x)$ based on

$$\widehat{\sigma}_\epsilon^2(x) = \widehat{\rho}_2(x) - 2\widehat{m}(x)\widehat{\rho}_1(x) + \widehat{m}^2(x),$$

where

$$\widehat{\rho}_l(x) = \frac{\sum_{i=1}^n K_i \delta_i \bar{G}_n^{-1}(T_i) T_i^l}{\sum_{i=1}^n K_i}, \quad \forall\, l \in \{1, 2\}.$$

Therefore, the approximate $(1 - \zeta/2) \times 100\%$ CI for $m(x)$ (the classical regression), for any $x \in \mathcal{F}$, is formulated as

$$\left[\widehat{m}(x) - z_{1-\zeta/2}\frac{\sqrt{\widehat{\beta}_2}\widehat{\sigma}_\epsilon(x)}{\widehat{\beta}_1}, \widehat{m}(x) + z_{1-\zeta/2}\frac{\sqrt{\widehat{\beta}_2}\widehat{\sigma}_\epsilon(x)}{\widehat{\beta}_1}\right]$$

In order to construct confidence bands (for both CKE and REE), we proceed by the following algorithm:

Step 1 Split our data into randomly chosen subsets: $(X_i, Y_i)_{i \in I}$ (training set) and $(X_j, Y_j)_{j \in J}$ (test set).
Step 2 Calculate the estimator $\widetilde{r}_n(X_i)$ for all $i \in I$ by using the training sample.
Step 3 For each X_j in the test sample, set $i^* := \arg\min_{i \in I} d(X_j, X_i)$.
Step 4 For all $j \in J$, define the confidence bands by means of

$$\left[\widetilde{r}_n(X_{i^*}) - z_{0.975}\widetilde{\sigma}_n(X_{i^*}), \widetilde{r}_n(X_{i^*}) + z_{0.975}\widetilde{\sigma}_n(X_{i^*})\right],$$

where $z_{0.975} \approx 1.96$ is the 97.5% quantile of a standard normal distribution.
Step 5 We present our results by plotting the extremities of the predicted values versus the true values and the confidence bands.

Figures 3 and 4 shows clearly a good behaviur of our estimator compared to the classical regression, with censorship rate (CR = 30%) and in the presence of outliers. In these figures, the solid black curve connects the true values. The dashed blue curves connect the lower and upper predicted values. The solid red curve connects the crossed points which give the predicted values.

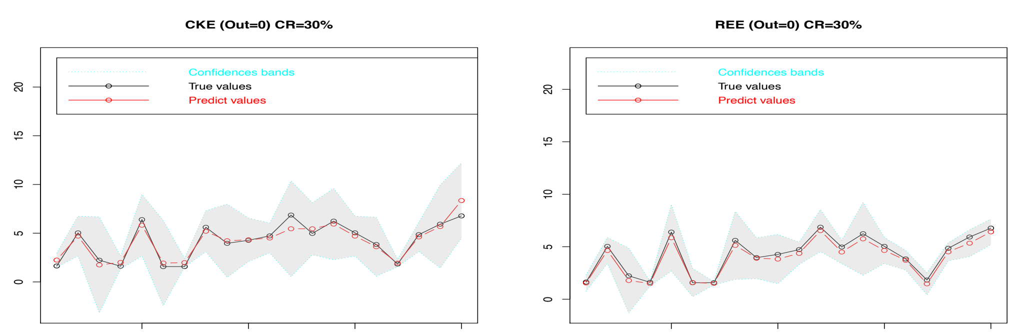

Figure 3. Extremities of the predicted values versus the true values and the confidence bands (simulation data without outliers).

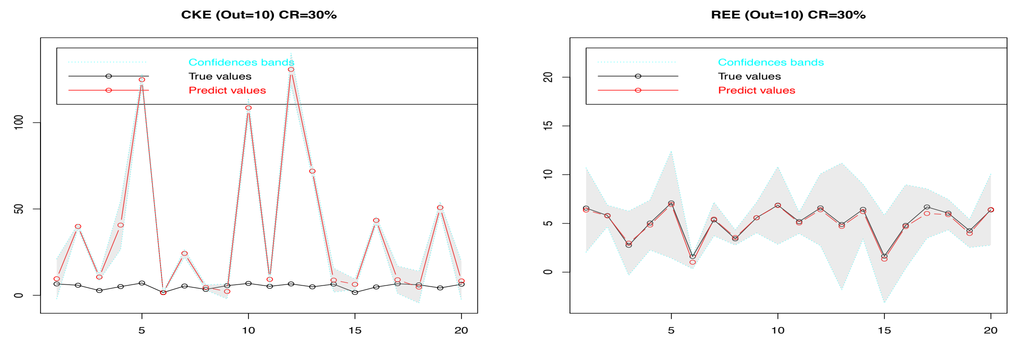

Figure 4. Extremities of the predicted values versus the true values and the confidence bands (simulation data in the presence of 10 outliers).

4.2 A real data application

First, we have acquired a large dataset, consisting of number of 8784 records, containing the hourly energy consumption for the year 2016 (measured in MWh), retrieved from the smart metering device of a commercial center type of consumer (a large hypermarket). We have also acquired a dataset containing the historical hourly meteorological data regarding the temperature (measured in Celsius degrees). These data were recorded by the meteorological sensors of a specialized institute for the year 2016, consisting in a number of 8784 records; see Pîrjan et al. (2017) and Mebsout et al. (2020) for more description on this data set.

Now, we are interested in the estimation of interval prediction of peak consumption of energy. For a fixed day i, let us denote by $(E_i(t_j))_{j=1,\ldots,24}$ the hourly measurements of some consumption of energy. The peak demand observed for the day i is defined as

$$P_i = \max_{j=1,\ldots,24} E_i\left(t_j\right).$$

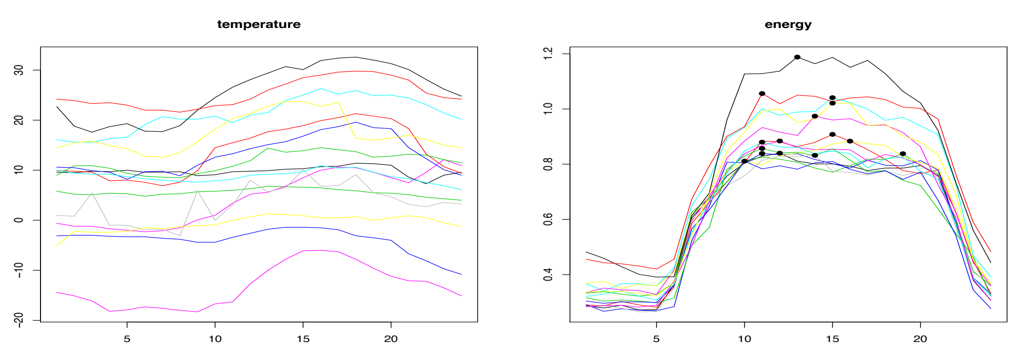

Figure 5. Sample of 15 daily temperature curves and the associated energy consumption curves.

It is well known that peak demand is very correlated with temperature measurements. Figure 5 provides a sample of 15 curves of hourly temperature measures and the associated electricity consumption curves. We split our sample of 366 days into a learning sample containing the first 300 days and a testing sample with the last 66 days. From the learning sample, we selected 30% of days within which we generated the censorship randomly. Figure 6 provides a sample of four censored daily load curves. For those days, we observe the electricity consumption until a certain time $t_c \in [1, 24]$, which corresponds to the time of censorship which is plotted in a dashed line in Figure 6. For a censored day, we define the censored random variable as

$$C_i = \max_{j=1,\dots,t_c} E_i(t_j).$$

Therefore, our sample is formed as follows $(X_i, Y_i, \delta_i)_{i=1,\dots,300}$, where $\delta_i = 1$ if $Y_i = P_i$ and $\delta_i = 0$ if $Y_i = C_i$. In order to introduce the outliers in this sample, we randomly multiplies by 10 some response variable of a number of observations.

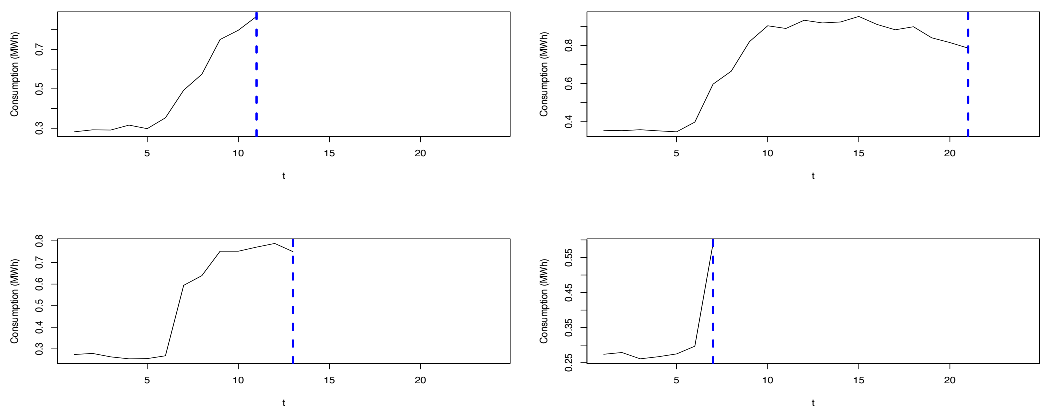

Figure 6. Sample of four censored daily load curves, the dashed line corresponds to the time of censorship t_c.

The selection of the bandwidth parameter is an important and basic problem in all kernel smoothing techniques. Another important point for ensuring a good behavior of the method is to use a semi-metric that is well adapted to the kind of data we have to deal with. Our data are based on the m eigen-functions of the empirical covariance operator associated with the m greatest eigenvalues (Ferraty and Vieu, 2006, Ch. 13). The estimators are obtained by choosing the optimal bandwidths by L^1 cross-validation method and the kernel K is the quadratic function defined by $K(x) = 3/2\,(1-x^2)\,\mathbb{1}_{[0,1]}$. The error used is expressed by

$$\text{MSE}_{\text{CKE}} = \frac{1}{66}\sum_{i=301}^{366}\left(Y_i - \widehat{m}(X_i)\right)^2 \quad \text{and} \quad \text{MSE}_{\text{REE}} = \frac{1}{66}\sum_{i=301}^{366}\left(Y_i - \widetilde{r}(X_i)\right)^2.$$

The results are given in Figure 7, where two curves corresponding to the observed values (black curve) the predicted values (dashed curve green for the classical regression and red for the relative one) are drawn. Clearly, Figure 7 shows the good behavior of our procedure. We observe that the relative approach gives better results than the classical regression approach ($\text{MSE}_{\text{CKE}} = 0.0883$ and $\text{MSE}_{\text{REE}} = 0.0034$).

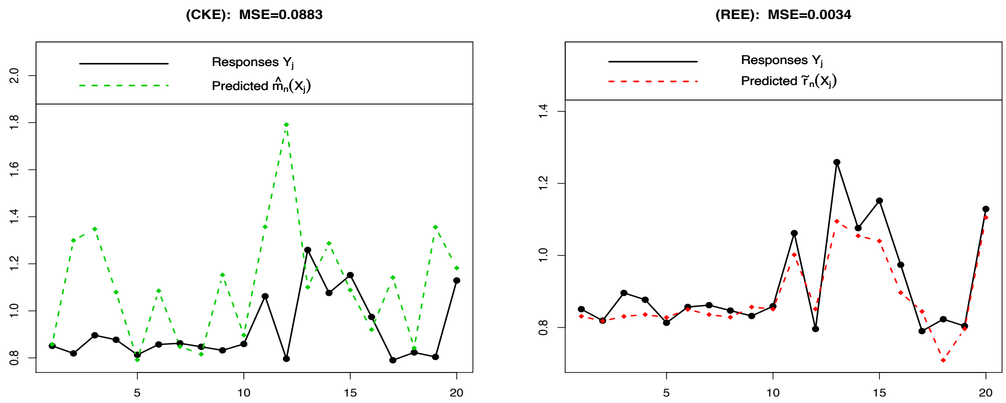

Figure 7. Prediction by classical and relative regression.

Now, we give in Table 2 the 90% predictive intervals of the concentrations for the peak load of the 20 last values in the sample test. This conclusion shows the good performance of our asymptotic normality.

Table 2. The 90% predictive intervals of the peak demand for the last 20 days.

True value	Predicted value	Predictive $CI_{90\%}$	True value	Predicted value	Predictive $CI_{90\%}$
0.851	0.8310	[0.6078, 1.0542]	1.062	1.0017	[0.8279, 1.1756]
0.819	0.8177	[0.7376, 0.8978]	0.796	0.8514	[0.7592, 0.9435]
0.896	0.8307	[0.7697, 0.8918]	1.259	1.0946	[0.9344, 1.2548]
0.877	0.8358	[0.4879, 1.1838]	1.076	1.0545	[0.8648, 1.2441]
0.813	0.8277	[0.4660, 1.1894]	1.152	1.0399	[0.9289, 1.1508]
0.857	0.8501	[0.5713, 1.1289]	0.974	0.8968	[0.7833, 1.0103]
0.862	0.8358	[0.7802, 0.8914]	0.790	0.8444	[0.7913, 0.8974]
0.847	0.8284	[0.3206, 1.3363]	0.823	0.7091	[0.0456, 1.3727]
0.832	0.8568	[0.7976, 0.9160]	0.804	0.7965	[0.6710, 0.9219]
0.859	0.8511	[0.7328, 0.9694]	1.129	1.1054	[0.8670, 1.3437]

Conclusions

In this paper, we have investigated the asymptotic properties of a nonparametric estimator of the relative error regression given a dependent functional explanatory variable, in the case of a scalar censored response. We have used the mean squared relative error as a loss function to construct a nonparametric estimator of the regression operator of these functional censored data. We have established the almost surely convergence and asymptotic normality of the proposed estimator. A simulation study and real data application were performed to support the theoretical results and to compare the quality of predictive performances of the relative error regression estimator than those obtained with standard kernel regression estimates. Our proposal provides interesting findings and is a tool that can be helpful to diverse practitioners. Our proposal has some limitations that open some doors for further research, which will be considered by the authors in future works.

Appendix

Proof of Theorem 3.2 From Equation (5), we have

$$|\widetilde{r}_n(x)-r(x)| \leq \frac{1}{|\widetilde{g}_{2,n}(x)|}\Big\{ |\widetilde{g}_{1,n}(x)-\widetilde{g}_1(x)| + |\widetilde{g}_1(x)-\mathbb{E}(\widetilde{g}_1(x))| + |\mathbb{E}(\widetilde{g}_1(x))-g_1(x)| \Big\} + \frac{|r(x)|}{|\widetilde{g}_{2,n}(x)|}\Big\{ |\widetilde{g}_{2,n}(x)-\widetilde{g}_2(x)| + |\widetilde{g}_2(x)-\mathbb{E}(\widetilde{g}_2(x))| + |\mathbb{E}(\widetilde{g}_2(x))-g_2(x)| \Big\}.$$

Therefore, Theorem 3.2's result is a consequence of the following intermediate results.

Lemma 4.1 Under hypotheses (H2)-(H5), we have

$$|\widetilde{g}_{l,n}(x)-\widetilde{g}_l(x)| = O_{\mathrm{AS}}\left(\sqrt{\frac{\log(\log(n))}{n}}\right),$$

for $l \in \{1,2\}$.

Lemma 4.2 Under hypotheses (H1)-(H3) and (H5), we get

$$|\mathbb{E}(\widetilde{g}_l(x))-g_l(x)| = O(h^{k_l}),$$

for $l \in \{1,2\}$.

Lemma 4.3 Under hypotheses (H1)-(H4) and (H6)-(H8), we obtain

$$|\widetilde{g}_l(x)-\mathbb{E}(\widetilde{g}_l(x))| = O_{\mathrm{ACC}}\left(\sqrt{\frac{\log(n)}{n\varphi_x(h)}}\right),$$

for $l \in \{1,2\}$.

Corollary 4.4 Under the hypotheses of Lemma 4.1 and 4.2, we have that

$$\text{there exists } \delta>0; \text{ such that } \sum_{n=1}^{\infty}\mathbb{P}\Big(|\widetilde{g}_{2,n}(x)|<\delta\Big) < \infty.$$

Let denote $K_i(x)$ by $K(d(x,X_i)/h)$.

Proof of Lemma 4.1
The proof is similar to Lemma 3.1 of Fetitah et al. (2020).

Proof of Lemma 4.2
For all $l=1,2$, we get that

$$\begin{aligned}|\mathbb{E}(\widetilde{g}_l(x))-g_l(x)| &= \left|\mathbb{E}\left(\frac{K_1(x)}{\mathbb{E}(K_1(x))}\mathbb{E}\left[\frac{\mathbb{E}(\mathbb{1}_{Y_1\leq C_1}|Y_1)Y_1^{-l}}{\bar{G}(Y_1)}|X_1\right]\right)-g_l(x)\right| \\ &= \frac{1}{\mathbb{E}(K_1(x))}\Big|\mathbb{E}\Big\{\Big[\mathbb{E}(Y_1^{-l}|X_1)-g_l(x)\Big]\mathbb{1}_{B(x,h)}(X_1)K_1(x)\Big\}\Big|.\end{aligned}$$

Then, by the Hölder hypothesis (H2) we obtain that

$$|g_l(X_1) - g_l(x)| \leq ch^{k_l}.$$

Thus,

$$|\mathbb{E}(\widetilde{g}_l(x)) - g_l(x)| \leq ch^{k_l}.$$

Proof of Lemma 4.3

For $l = 1, 2$ we put

$$\Delta_i(x) = \frac{\delta_i T_i^{-l}}{\bar{G}(T_i)} K\left(\frac{d(x, X_i)}{h}\right) - \mathbb{E}\left[\frac{\delta_i T_i^{-l}}{\bar{G}(T_i)} K\left(\frac{d(x, X_i)}{h}\right)\right].$$

The use of the Fuk-Nagaev inequality (Rio, 1999, p. 87, 6.19b), which is based on

$$\begin{aligned} S_n^2 &= \sum_{i=1}^{n}\sum_{j=1}^{n} |\mathrm{Cov}\,(\Delta_i(x), \Delta_j(x))| \\ &= \sum_{i\neq j} |\mathrm{Cov}\,(\Delta_i(x), \Delta_j(x))| + n\,\mathrm{Var}\,(\Delta_1(x)). \end{aligned}$$

Now, by using (H5), we get

$$\begin{aligned} \mathrm{Var}\,(\Delta_1(x)) &\leq \mathbb{E}\left[\frac{\delta_1 Y_1^{-2l}}{\bar{G}^2(Y_1)} K_1^2(x)\right] + \mathbb{E}^2\left[\frac{\delta_1 Y_1^{-l}}{\bar{G}(Y_1)} K_1(x)\right] \\ &\leq \mathbb{E}\left[K_1^2(x)\mathbb{E}\left(\frac{\mathbb{E}(\mathbb{1}_{Y_1\leq C_1}|Y_1)Y_1^{-2l}}{\bar{G}^2(Y_1)}|X_1\right)\right] \\ &\quad +\mathbb{E}^2\left[K_1(x)\mathbb{E}\left(\frac{\mathbb{E}(\mathbb{1}_{Y_1\leq C_1}|Y_1)Y_1^{-l}}{\bar{G}(Y_1)}|X_1\right)\right] \\ &\leq \frac{c}{\bar{G}(\tau_F)}\mathbb{E}\big[K_1^2(x)\mathbb{E}(Y_1^{-2l}|X_1)\big] + \mathbb{E}^2\big[K_1(x)\mathbb{E}(Y_1^{-l}|X_1)\big] \\ &\leq \frac{c}{\bar{G}(\tau_F)}\mathbb{E}\big[K_1^2(x)\big] + c\varphi_x^2(h) \\ &\leq c\big(\varphi_x(h) + \varphi_x^2(h)\big). \end{aligned}$$

In addition, for $i \neq j$, we have

$$\begin{aligned} |\mathrm{Cov}\,(\Delta_i(x), \Delta_j(x))| &= |\mathbb{E}\,(\Delta_i(x)\Delta_j(x))| \\ &\leq c\,|\mathbb{E}\,(K_i(x)K_j(x)) + \mathbb{E}\,(K_i(x))\,\mathbb{E}\,(K_j(x))|. \end{aligned}$$

Then, following Masry (1986), we define the sets given by $E_1 = \{(i, j),$ such that $1 \leq |i-j| \leq \nu_n\}$ and $E_2 = \{(i, j)$ such that $\nu_n + 1 \leq |i - j| \leq n\}$, where $\nu_n \to \infty$ as $n \to \infty$. Then, we can write $\sum_{i\neq j} |\mathrm{Cov}\,(\Delta_i(x), \Delta_j(x))| = J_{1,n} + J_{2,n}$, where $J_{1,n}$ and $J_{2,n}$ are the sums of the covariances over E_1 and E_2 respectively. Therefore, under (H7), we get

$$\begin{aligned} J_{1,n} &= \sum_{E_1} |\mathrm{Cov}\,(\Delta_i(x), \Delta_j(x))| \leq c \sum_{E_1} \left|\mathbb{E}(K_i(x)K_j(x)) + \mathbb{E}(K_1(x))^2\right| \\ &\leq c \sum_{E_1} \left|\mathbb{P}\big(\,(X_i, X_j) \in B(x,h) \times B(x,h)\big) + \varphi_x(h)^2\right| \\ &\leq c n \nu_n \varphi_x(h) \left[\left(\frac{\varphi_x(h)}{n}\right)^{\frac{1}{a}} + \varphi_x(h)\right]. \end{aligned}$$

For the second term, we use the modified Davydov covariance inequality for mixing processes (Rio, 1999, p.10). Then, we have

$$\forall i \neq j, |\mathrm{Cov}\,(\Delta_i(x), \Delta_j(x))| \leq c\alpha(|i-j|).$$

Thus, we get by (H6) that

$$J_{2,n} \leq \sum_{E_2} |\mathrm{Cov}\,(K_i(x), K_j(x))| \leq n^2 \nu_n^{-a}.$$

Hence, for $\nu_n = (\varphi_x(h)/n)^{-1/a}$, we have $\sum_{i \neq j} |\mathrm{Cov}\,(\Delta_i(x), \Delta_j(x))| = O(n\varphi_x(h))$. Consequently, combining previous result, we obtain

$$S_n^2 = O(n\varphi_x(h)). \tag{6}$$

Using the Fuk-Nagaev inequality, we get, for all $l = 1, 2$, $\varepsilon > 0$ and $r > 1$, that

$$\begin{aligned} \mathbb{P}\left[\left|\mathbb{E}[\widetilde{g}_l(x)] - \widetilde{g}_l(x)\right| > \varepsilon\right] &= \mathbb{P}\left[\left|\frac{1}{n\mathbb{E}(K_1(x))} \sum_{i=1}^{n} \Delta_i(x)\right| > \varepsilon\right] \\ &= \mathbb{P}\left[\left|\sum_{i=1}^{n} \Delta_i(x)\right| > \varepsilon n \mathbb{E}(K_1(x))\right] \\ &\leq c\left\{\left(1 + \frac{\varepsilon^2 n^2 \mathbb{E}(K_1(x))^2}{r S_n^2}\right)^{-r/2} + n r^{-1}\left(\frac{r}{\varepsilon n \mathbb{E}(K_1(x))}\right)^{a+1}\right\} \\ &\leq c(A_1 + A_2), \end{aligned}$$

where

$$A_1 = \left(1 + \frac{\varepsilon^2 n^2 \left(\mathbb{E}\left[K_1(x)\right]\right)^2}{r S_n^2}\right)^{-r/2} \quad \text{and} \quad A_2 = n r^{-1} \left(\frac{r}{\varepsilon n \mathbb{E}\left[K_1(x)\right]}\right)^{a+1}.$$

Therefore, by Equation (6) and putting

$$\varepsilon = \varepsilon_0 \sqrt{\frac{\log(n)}{n\varphi_x(h)}} \quad \text{and} \quad r = (\log(n))^2,$$

it follow that $A_2 \leq c n^{1-(a+1)/2} \varphi_x(h)^{-(a+1)/2} (\log(n))^{(3a-1)/2}$. Next, using the left side of (H8), we obtain $A_2 \leq c n^{-1-\eta(a+1)/2} (\log(n))^{(3a-1)/2}$. Hence, it exists some real $\nu > 0$ such that

$$A_2 \leq cn^{-1-\nu}. \tag{7}$$

Because of $r = (\log(n))^2$, we show that

$$A_1 \leq \left(1 + \frac{\varepsilon_0^2}{\log(n)}\right)^{-\frac{(\log(n))^2}{2}} = e^{-\frac{(\log(n))^2}{2}} \log\left(1 + \frac{\varepsilon_0^2}{\log(n)}\right).$$

Using the fact that $\log(1+x) = x - x^2/2 + o(x^2)$, when $x \to 0$, we get

$$A_1 \leq e^{-\frac{\varepsilon_0^2 \log(n)}{2}} = n^{-\frac{\varepsilon_0^2}{2}}.$$

The last result allows us to get directly that there exist some ε_0 and some ν' such that

$$A_1 \leq cn^{-1-\nu'}. \tag{8}$$

Therefore, by the results of Equations (8) and (7), we have

$$\sum_{n\geq 1} \mathbb{P}\left[\left|\mathbb{E}[\widetilde{g}_l(x)] - \widetilde{g}_l(x)\right| > \varepsilon_0 \sqrt{\frac{\log(n)}{n\varphi_x(h)}}\right] < \infty.$$

Proof of Corollary 4.4
The proof of this Corollary is analogous to Corollary 2 of Demongeot et al. (2016).

Proof of Theorem 3.3
From Equation (5), we adopt the decomposition stated as

$$\widetilde{r}_n(x) - r(x) = \widetilde{r}_n(x) - \widetilde{r}(x) + \widetilde{r}(x) - r(x) =: I_{1n}(x) + I_{2n}(x)$$

where

$$I_{1n}(x) =: \widetilde{r}_n(x) - \widetilde{r}(x) \quad \text{and} \quad I_{2n}(x) =: \widetilde{r}(x) - r(x).$$

The proof is derived by showing first that $I_{1n}(x)$ is negligible whereas $I_{2n}(x)$ is asymptotically normal distributed.

From Lemma 4.1 and Corollary 4.4, we deduce that

$$I_{1n}(x) \xrightarrow{\mathbb{P}} 0. \tag{9}$$

Now, we can write that

$$I_{2n}(x) = \frac{1}{\widetilde{g}_2(x)}\Big[D_n + A_n\Big(\mathbb{E}\left[\widetilde{g}_2(x)\right] - \widetilde{g}_2(x)\Big)\Big] + A_n, \tag{10}$$

where

$$A_n = \frac{1}{\mathbb{E}\left[\widetilde{g}_2(x)\right] g_2(x)} \Big\{ \mathbb{E}\left[\widetilde{g}_1(x)\right] g_2(x) - \mathbb{E}\left[\widetilde{g}_2(x)\right] g_1(x) \Big\}$$

$$D_n = \frac{1}{g_2(x)} \Big[V_{1n}\left(x\right) g_2(x) - V_{2n}\left(x\right) g_1(x) \Big],$$

whith $V_{ln}\left(x\right) = \widetilde{g}_l(x) - \mathbb{E}\left[\widetilde{g}_l(x)\right]$, for $l = 1, 2$.
Then, it follows from Equation (10) that

$$\begin{aligned} \widetilde{r}(x) - r(x) - A_n &= \frac{1}{\widetilde{g}_2(x)} \Big[D_n + A_n \Big(\mathbb{E}\left[\widetilde{g}_2(x)\right] - \widetilde{g}_2(x) \Big) \Big] \\ &=: \frac{D_n - A_n V_{2n}(x)}{\widetilde{g}_2(x)}. \end{aligned}$$

Consequently, the proof of Theorem 3.3 can be deduced from the convergence in Equation (9) and the following intermediate results (cf. Lemmas 4.5, 4.6 and 4.7).

Lemma 4.5 Under hypotheses of Theorem 3.3, we have

$$\left(\frac{n\varphi_x(h)}{g_2^2(x)\sigma^2(x)} \right)^{1/2} \Big(\Big[\widetilde{g}_1(x) - \mathbb{E}\left[\widetilde{g}_1(x)\right] \Big] g_2(x) - \Big[\widetilde{g}_2(x) - \mathbb{E}\left[\widetilde{g}_2(x)\right] \Big] g_1(x) \Big) \xrightarrow{\mathcal{D}} \mathcal{N}(0,1).$$

Lemma 4.6 Under hypotheses of Theorem 3.3, we obtain

$$A_n = hB_n + o(h).$$

Lemma 4.7 Under hypotheses of Theorem 3.3, we obtain

$$\widetilde{g}_2(x) \xrightarrow{\mathbb{P}} g_2(x),$$

and

$$\left(\frac{n\varphi_x(h)}{g_2^2(x)\sigma^2(x)} \right)^{1/2} A_n \left(\mathbb{E}[\widetilde{g}_2(x)] - \widetilde{g}_2(x) \right) \xrightarrow{\mathbb{P}} 0.$$

Proof of Lemma 4.5
It is easy to see that

$$\sqrt{n\varphi_x(h)} \Big[\Big(\widetilde{g}_2(x) - \mathbb{E}\left[\widetilde{g}_2(x)\right] \Big) g_1(x) - \Big(\widetilde{g}_1(x) - \mathbb{E}\left[\widetilde{g}_1(x)\right] \Big) g_2(x) \Big] = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} L_i(x),$$

where

$$L_i(x) := \frac{\sqrt{\varphi_x(h)}}{\mathbb{E}\left[K_1\right]} \left\{ \frac{\delta_i}{\bar{G}\left(T_i\right)} K_i \left(g_1(x) T_i^{-2} - g_2(x) T_i^{-1} \right) - \mathbb{E}\left[\frac{\delta_i}{\bar{G}\left(T_i\right)} K_i \left(g_1(x) T_i^{-2} - g_2(x) T_i^{-1} \right) \right] \right\}.$$

The proof of this lemma is based on the central limit theorem of Doukhan et al. (1994). We have then to consider the asymptotic behavior of the variance term and the following assumption

$$\int_0^1 \alpha^{-1}(u)\,(Q_{L_1}(u))^2\,\mathrm{d}u < +\infty,$$

where Q_{L_1} is the upper tail quantile function defined by

$$Q_{L_1}(u) = \inf\{t \geq 0 : \mathbb{P}(L_1 > t) \leq u\}$$

and $\alpha^{-1}(u) = \sum_{n\in\mathbb{N}} \mathbb{1}_{u<\alpha_n}$. Clearly,

$$\begin{aligned}
\operatorname{Var}\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n} L_i(x)\right) &= n\varphi_x(h)\operatorname{Var}\left(\frac{g_1(x)}{n\mathbb{E}[K_1]}\sum_{i=1}^{n}\frac{\delta_i}{\bar{G}(T_i)}K_iT_i^{-2} - \frac{g_2(x)}{n\mathbb{E}[K_1]}\sum_{i=1}^{n}\frac{\delta_i}{\bar{G}(T_i)}K_iT_i^{-1}\right)\\
&= n\varphi_x(h)\Big(\operatorname{Var}[\widetilde{g}_1(x)]\,g_2^2(x) + \operatorname{Var}[\widetilde{g}_2(x)]\,g_1^2(x)\\
&\quad -2g_1(x)g_2(x)\operatorname{Cov}[\widetilde{g}_1(x),\widetilde{g}_2(x)]\Big).
\end{aligned}$$

By definition of $\widetilde{g}_l(x)$ for $l = 1; 2$, we have

$$\begin{aligned}
n\varphi_x(h)\operatorname{Var}[\widetilde{g}_l(x)] &= \frac{\varphi_x(h)}{(\mathbb{E}[K_1])^2}\operatorname{Var}\left[\frac{\delta_1}{\bar{G}(T_1)}K_1T_1^{-l}\right]\\
&\quad + \frac{\varphi_x(h)}{n(\mathbb{E}[K_1])^2}\sum_{i=1}^{n}\sum_{\substack{j=1\\|i-j|>0}}^{n}\operatorname{Cov}\left[\frac{\delta_i}{\bar{G}(T_i)}K_iT_i^{-l}, \frac{\delta_j}{\bar{G}(T_j)}K_jT_j^{-l}\right]\\
&= \frac{\varphi_x(h)}{(\mathbb{E}[K_1])^2}J_{1,1} + \frac{\varphi_x(h)}{n(\mathbb{E}[K_1])^2}J_{2,n}
\end{aligned}$$

where

$$\begin{aligned}
J_{1,1} &= \operatorname{Var}\left[\frac{\delta_1}{\bar{G}(T_1)}K_1T_1^{-l}\right],\\
J_{2,n} &= \sum_{i=1}^{n}\sum_{\substack{j=1\\|i-j|>0}}^{n}\operatorname{Cov}\left[\frac{\delta_i}{\bar{G}(T_i)}K_iT_i^{-l}, \frac{\delta_j}{\bar{G}(T_j)}K_jT_j^{-l}\right].
\end{aligned}$$

By conditioning on the random variable X_1, by the same ideas in the proof of lemma 4.2, Lemma 4 in Ferraty et al. (2007) and by using hypotheses (H5), (N1) and (N4), we get

$$\begin{aligned}
\mathbb{E}\left[\left(\frac{\delta_1Y_1^{-l}}{\bar{G}(Y_1)}\right)^2K_1^2(x)\right] &= \mathbb{E}\left[K_1^2(x)\mathbb{E}\left(\frac{\mathbb{E}(\mathbb{1}_{Y_1\leq C_1}|Y_1)Y_1^{-2l}}{\bar{G}^2(Y_1)}|X_1\right)\right]\\
&= \left[\mathbb{E}\left(\frac{Y_1^{-2l}}{\bar{G}(Y_1)}|X_1 = x\right) + o(1)\right]\mathbb{E}\left[K_1^2(x)\right]\\
&= \varphi_x(h)\mathbb{E}\left[\frac{Y_1^{-2l}}{\bar{G}(Y_1)}|X_1 = x\right]\left(K^2(1) - \int_0^1\left(K^2(s)\right)'\chi_x(u)\mathrm{d}u\right) + o(\varphi_x(h))
\end{aligned}$$

and

$$\mathbb{E}\left(\frac{\delta_1}{\bar{G}(Y_1)}Y_1^{-l}K_1\right) = O(\varphi_x(h)).$$

Thus:

$$\operatorname{Var}\left[\frac{\delta_1}{\bar{G}(T_1)}T_1^{-l}K_1\right] = \varphi_x(h)\mathbb{E}\left[\bar{G}^{-1}(Y_1)Y_1^{-2l}|X_1 = x\right]\left(K^2(1) - \int_0^1 \left(K^2(s)\right)' \chi_x(u)\mathrm{d}u\right) + O\left(\varphi_x^2(h)\right).$$

We obtain

$$\frac{\varphi_x(h)}{\left(\mathbb{E}\left[K_1\right]\right)^2}J_{1,1} \to \frac{q_{2l}(x)\beta_2}{\beta_1^2}. \tag{11}$$

Let us turn to $J_{2,n}$, for this we use the technique of Masry (1986). We define the same sets E_1 and E_2 in the proof of Lemma 4.3. Let $J_{2,n}^1$ and $J_{2,n}^2$ be the sums of covariances over E_1 and E_2 respectively. On the one hand, we have

$$J_{2,n}^1 = \sum_{E_1}\left|\operatorname{Cov}\left[\frac{\delta_i}{\bar{G}\left(T_i\right)}K_iT_i^{-l}, \frac{\delta_j}{\bar{G}\left(T_j\right)}K_jT_j^{-l}\right]\right| \leq C\sum_{E_1}\left|\mathbb{E}\left[K_iK_j\right] - \mathbb{E}\left[K_i\right]\mathbb{E}\left[K_j\right]\right|.$$

Because of the assumptions of Lemma 4.3 we can write

$$J_{2,n}^1 \leq cn\nu_n\varphi_x(h)\left(\left(\frac{\varphi_x(h)}{n}\right)^{\frac{1}{a}} + \varphi_x(h)\right).$$

Hence, for the summation over E_2, we use the Davydov-Rio inequality (Rio, 1999, p. 87), for mixing processes. This leads, for all $i \neq j$, to

$$\left|\operatorname{Cov}\left(K_i, K_j\right)\right| \leq c\alpha(|i-j|),$$

Therefore,

$$\sum_{E_2}\left|\operatorname{Cov}\left(K_i, K_j\right)\right| \leq n^2\nu_n^{-a}.$$

The choice $\nu_n = 1/[\varphi_x(h)\log(n)]$, motivated by the upper bound in (H8), permits to get

$$\sum_{i\neq j}^{n}\operatorname{Cov}\left(K_i, K_j\right) = o\left(n\varphi_x(h)\right),$$

then

$$\frac{\varphi_x(h)}{n\left(\mathbb{E}\left[K_1\right]\right)^2}J_{2,n} = o(1) \text{ as } n \longrightarrow \infty. \tag{12}$$

Thanks to Equations (11) and (12), we have

$$n\varphi_x(h)\,\mathrm{Var}\left(\widetilde{g}_l(x)\right) \longrightarrow \frac{\beta_2 q_{2l}(x)}{\beta_1^2} \text{ as } n \longrightarrow \infty. \tag{13}$$

Concerning the covariance term, we follow the same steps as for the variance given in Equation (13) then we get:

$$n\varphi_x(h)\,\mathrm{Cov}\left(\widetilde{g}_1(x), \widetilde{g}_2(x)\right) \longrightarrow \frac{\beta_2 q_3(x)}{\beta_1^2} \text{ as } n \longrightarrow \infty. \tag{14}$$

Let us now prove the claimed result. Clearly, the function Q_{L_1} is nonincreasing, then

$$\sum_{n=1}^{\infty} \int_0^{\alpha_n} \left[Q_{L_1}(u)\right]^2 \mathrm{d}u \leq \sum_{n=1}^{\infty} \alpha_n Q_{L_1}^2(0).$$

By hypotheses (H1), (H3) and (H5) we can write

$$c\frac{1}{\sqrt{\varphi_x(h)}} \leq |L_1| \leq c'\frac{1}{\sqrt{\varphi_x(h)}}.$$

Then,

$$Q_{L_1}(0) \leq c'\frac{1}{\sqrt{\varphi_x(h)}}.$$

Therefore, we have

$$\sum_{n=1}^{\infty} \int_0^{\alpha_n} \left[Q_{L_1}(u)\right]^2 \mathrm{d}u \leq \sum_{n=1}^{\infty} \alpha_n \left(\varphi_x(h)\right)^{-1}.$$

It follows from (H7) and (H8) that

$$\sum_{n=1}^{\infty} \int_0^{\alpha_n} \left[Q_{L_1}(u)\right]^2 \mathrm{d}u < \infty. \tag{15}$$

From Equations (13), (14) and by noting

$$\sigma^2(x) = \frac{\Big(q_2(x) - 2r(x)q_3(x) + r^2(x)q_4(x)\Big)\beta_2}{\beta_1^2},$$

we conclude that

$$\mathrm{Var}\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n} L_i(x)\right) \longrightarrow \sigma^2(x) \text{ as } n \longrightarrow \infty. \tag{16}$$

Now, the lemma can be easily deduced from Equations (15), (16) and the central limit theorem of Doukhan et al. (1994) as

$$\frac{1}{\sqrt{ng_2^2\sigma^2(x)}}\sum_{i=1}^{n}L_i(x)=\left(\frac{n\varphi_x(h)}{g_2^2(x)\sigma^2(x)}\right)^{1/2}$$
$$\times\left(\left[\widetilde{g}_1(x)-\mathbb{E}\left[\widetilde{g}_1(x)\right]\right]g_2(x)-\left[\widetilde{g}_2(x)-\mathbb{E}\left[\widetilde{g}_2(x)\right]\right]g_1(x)\right)\xrightarrow{\mathcal{D}}\mathcal{N}(0,1).$$

Proof of Lemma 4.6
As in Ferraty et al. (2007) we show that:

$$\mathbb{E}[\widetilde{r}_n(x)]=\frac{\mathbb{E}\left[\widetilde{g}_1(x)\right]}{\mathbb{E}\left[\widetilde{g}_2(x)\right]}+O\left(\frac{1}{n\varphi_x(h)}\right).$$

So, it suffices to evaluate $\mathbb{E}[\widetilde{g}_l(x)]$ for $l\in\{1,2\}$, we obtain

$$\begin{aligned}\mathbb{E}\left[\widetilde{g}_1(x)\right]&=\frac{1}{\mathbb{E}\left[K_1\right]}\mathbb{E}\Big(K_1(x)\mathbb{E}\left[Y_1^{-l}|X_1\right]\Big)\\&=\frac{1}{\mathbb{E}\left[K_1\right]}\Big(g_l(x)E\left[K_1\right]+E\Big[K_1E\Big(g_l\left(X_1\right)-g_l(x)|d\left(X_1,x\right)\Big)\Big]\Big)\\&=g_l(x)+\frac{E\Big[K_1\Big(\Psi_l\left(d\left(X_1,x\right)\right)\Big)\Big]}{\mathbb{E}\left[K_1\right]}\\&=g_l(x)+\frac{\int_0^1K(t)\Psi_l(ht)\mathrm{d}\mathbb{P}^{d(x,X)/h}(t)}{\int_0^1K(t)\mathrm{d}\mathbb{P}^{d(x,X)/h}(t)}.\end{aligned}$$

By using the first-order Taylor expansion for Ψ_l around 0, where $\Psi_l(0)=0$, we have

$$E\left[\widetilde{g}_l(x)\right]=g_l(x)+h\Psi_l'(0)\left[\frac{\int_0^1tK(t)\mathrm{d}\mathbb{P}^{d(x,X)/h}(t)}{\int_0^1K(t)\mathrm{d}\mathbb{P}^{d(x,X)/h}(t)}\right]+o(h).$$

According to *Lemma 2* of Ferraty et al. (2007) we get, under (N1)

$$\frac{\int_0^1tK(t)\mathrm{d}\mathbb{P}^{d(x,X)/h}(t)}{\int_0^1K(t)\mathrm{d}\mathbb{P}^{d(x,X)/h}(t)}\longrightarrow\frac{\beta_0}{\beta_1}\text{ and }\int_0^1K(t)\mathrm{d}\mathbb{P}^{d(x,X)/h}(t)\longrightarrow\beta_1.$$

Consequently

$$E\left[\widetilde{g}_l(x)\right]=g_l(x)+h\Psi_l'(0)\frac{\beta_0}{\beta_1}+o(h)$$

then we deduce that:

$$A_n=\frac{\mathbb{E}\left[\widetilde{g}_1(x)\right]}{\mathbb{E}\left[\widetilde{g}_2(x)\right]}-r(x)=hB_n+o(h).$$

Proof of Lemma 4.7
The same idea in the proof of Lemma 3.6 of Fetitah et al. (2020).

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