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Research Paper

Estimating Ordered Scale Parameters of Two Exponential Populations With a Common Location Under Type-II Censoring

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Abstract

Estimation of ordered scale parameters of two exponential populations has been considered when the location parameter is common using type-II censored samples. Sufficient conditions for improving affine and scale equivariant estimators have been obtained when the scale parameters are ordered. A simulation study has been done in order to numerically compare the risk values of all the proposed estimators.

Keywords: Affine equivariant estimators · Inadmissibility · Maximum likelihood estimator · Ordered parameters · Quadratic loss function · Relative risk performance · Type-II censoring · Uniformly minimum variance unbiased estimator.

1. Introduction

Suppose type-II censored samples are available from two exponential populations with a common location and possibly different scale parameters. More specifically, let $X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(r)}$ $(2 \leq r \leq m)$ and $Y_{(1)} \leq Y_{(2)} \leq \cdots \leq Y_{(s)}$ $(2 \leq s \leq n)$ be ordered observations taken from two random samples of sizes m and n which follow $Ex(\mu, \sigma_1)$ and $Ex(\mu, \sigma_2)$ respectively. This type of data are known as type-II right censored data. Here $Ex(\mu, \sigma_i)$ denotes the exponential population with location parameter 'μ' and scale parameter σ_i, $i = 1, 2$. The probability density function of $Ex(\mu, \sigma_i)$ is given by

$$f(t, \mu, \sigma_i) = \frac{1}{\sigma_i} \exp\Big\{ -\Big(\frac{t-\mu}{\sigma_i}\Big)\Big\}, \quad t > \mu, \sigma_i > 0, -\infty < \mu < \infty; \quad i = 1, 2. \qquad (1.1)$$

The parameter 'μ' which is common to both populations is known as the location parameter (equivalently minimum guarantee time) and the σ_i's are known as the scale parameters (equivalently residual life times). The problem is to estimate the vector parameter $\underset{\sim}{\sigma} = (\sigma_1, \sigma_2)$ under the assumption that $\sigma_1 \leq \sigma_2$ using a decision theoretic approach. The

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loss function is taken as

$$L(\hat{\underset{\sim}{d}}, \underset{\sim}{\sigma}) = \sum_{i=1}^{2} \left(\frac{d_i - \sigma_i}{\sigma_i} \right)^2, \tag{1.2}$$

where $\hat{\underset{\sim}{d}} = (d_1, d_2)$ is an estimator for $\underset{\sim}{\sigma} = (\sigma_1, \sigma_2)$. The performance of an estimator will be evaluated using the risk function defined as

$$R(\hat{\underset{\sim}{d}}, \underset{\sim}{\sigma}) = E_{\underset{\sim}{\sigma}}\{L(\hat{\underset{\sim}{d}}, \underset{\sim}{\sigma})\}. \tag{1.3}$$

The problem of estimating parameters of exponential populations using censored samples is not new in the literature and has been extensively studied by several authors in the recent past. The censoring schemes available are type-I (number of failures are random but time is fixed), type-II (time is random but number of failures is fixed) or random censoring where both the number of failures and the time may be random. Most of the results available in these directions are based on one population only. For some recent results and review on estimation of parameters of an exponential population using these types of censoring schemes we refer to Lawless (1982) and Johnson et al. (2004). Some applications of these types of censoring schemes have also been discussed in Lawless (1982). Recently, some advanced censoring schemes have been developed by several authors which are more or less generalizations of these three types of censoring schemes. We note that, the type-II right censoring scheme is a particular case of progressive type-II censoring. For a detailed review and recent updates on estimating parameters of an exponential population using progressive type-II censored samples we refer to Madi (2010), Wang et al. (2010) and Balakrishnan and Cramer (2014).

However, a less attention has been paid to estimating the parameters when more than one exponential population is available. For example, Chiou and Cohen (1984) considered the estimation of the common location parameter of two exponential populations using type-II right censored data when the scale parameters are unknown. Elfessi and Pal (1991) considered the estimation of common scale and the location parameters of $k(\geq 2)$ exponential populations using type-II right censored data. Yike and Heliang (1999) considered the Bayesian estimation of ordered location parameters of two exponential populations under a multiple type-II censoring scheme. Tripathy (2015) obtained classes of equivariant estimators and derived some inadmissibility results for estimating the common location parameter of two exponential populations using type-II right censored data. Herein, we consider the model that has been previously considered by Chiou and Cohen (1984) and Tripathy (2015) and estimate the vector parameter $\underset{\sim}{\sigma} = (\sigma_1, \sigma_2)$, when ordering of the scale parameters is known in advance, that is, $\sigma_1 \leq \sigma_2$.

The model we consider in this paper has applications in industry, business, medical research in the study of reliability, life testing and survival analysis. For example, two new brands of electronic devices say brand A (which uses traditional technology) and brand B (which uses modern technology), having $m(\geq 2)$ and $n(\geq 2)$ units each are placed for life testing. The experimenter could observe only r $(\leq m)$ and s $(\leq n)$ number of failures from brand A and brand B respectively, due to some constraints like time and cost. It may be noted that, the lifetimes of each unit from the two brands are random and follow exponential distribution. It is also expected that the minimum guarantee time (μ) for both brands are the same due to market competition, whereas the residual life time (σ_1) of brand A can not exceed the residual life time (σ_2) of brand B. Under this situation one may be interested in drawing inference on the vector parameter $\underset{\sim}{\sigma} = (\sigma_1, \sigma_2)$. For some more examples we refer to Jana and Kumar (2015), and Barlow et al. (1972).

The problem of estimating the ordered parameters of various distribution functions has been studied by several researchers in the recent past, when full samples ($r = m$, $s = n$) are available. For some results on estimation of ordered parameters of two or more exponential populations we refer to Misra and Singh (1994), Jin and Pal (1991), Vijayasree et al. (1995), and Jana and Kumar (2015). Some work has been done in estimating the ordered parameters (means or variances) when the underlying distribution is normal. We refer to Chang et al. (2012) and Tripathy and Kumar (2011) for some results on estimating ordered parameters of normal populations.

It should be noted that, for full sample case ($r = m$, and $s = n$) Jana and Kumar (2015) considered the componentwise estimation of ordered scale parameters of two exponential populations when the location parameter is common. In this paper, we consider the simultaneous estimation of ordered scale parameters, that is, the vector $\underset{\sim}{\sigma} = (\sigma_1, \sigma_2) : \sigma_1 \leq \sigma_2$ using type-II right censored samples from two exponential populations. The rest of the paper is organized as follows. Section 2 introduces the MLE and the UMVUE without considering order restriction on the scale parameters. Then under order restriction on the scale parameters we derive the restricted maximum likelihood estimator for $\underset{\sim}{\sigma}$. In Section 3, we obtain classes of equivariant estimators and prove some inadmissibility results in these classes. Using these results we obtain improved estimators which dominate the MLE and the UMVUE with respect to the risk function (1.3). In Section 4, a detailed simulation study has been carried out to numerically compare the relative risk performances of all the proposed estimators and recommendations have been made regarding their use. The paper concludes with some remarks (Section 5).

2. Some Basic Results

In this section, we consider the model (1.1) and obtain some basic estimators for the vector parameter $\underset{\sim}{\sigma} = (\sigma_1, \sigma_2)$ assuming that $\sigma_1 \leq \sigma_2$. To be very specific, let $X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(r)}$, $(2 \leq r \leq m)$ be the r smallest ordered observations taken from a random sample of size $m(\geq 2)$ which follows $Ex(\mu, \sigma_1)$. Likewise let $Y_{(1)} \leq Y_{(2)} \leq \cdots \leq Y_{(s)}$, $(2 \leq s \leq n)$ be the s smallest ordered observations taken from a random sample of size $n(\geq 2)$ following $Ex(\mu, \sigma_2)$. We assume that these two samples have been drawn independently from two populations. Let us denote $Z = \min(X_{(1)}, Y_{(1)})$, $V_x = U_x - Z$, $V_y = U_y - Z$, where $U_x = [\sum_{i=1}^{r} X_{(i)} + (m-r)X_{(r)}]/m$ and $U_y = [\sum_{i=1}^{s} Y_{(i)} + (n-s)Y_{(s)}]/n$. The complete and sufficient statistics for this problem is given by (Z, V_x, V_y). The joint probability density function of (Z, V_x, V_y) is given by

$$f_{V_x,V_y,Z}(v_x, v_y, z) = \frac{m^r n^s}{\sigma_1^r \sigma_2^s}\Big[\frac{v_x^{r-1} v_y^{s-2}}{\Gamma r \Gamma(s-1)} + \frac{v_x^{r-2} v_y^{s-1}}{\Gamma s \Gamma(r-1)}\Big] \exp\Big\{-\frac{m}{\sigma_1}(v_x + z - \mu) - \frac{n}{\sigma_2}(v_y + z - \mu)\Big\},$$
$$v_x > 0, v_y > 0, z > \mu.$$

The statistic Z is independent of (V_x, V_y). In the next lines to follow, when we say the MLE (the UMVUE) of the vector parameter $\underset{\sim}{\sigma} = (\sigma_1, \sigma_2)$ we mean "the collection of the MLEs (the UMVUEs) for each component σ_i and put together to form the vector".

When there is no order restriction among the scale parameters σ_1 and σ_2 the MLE of $\underset{\sim}{\sigma} = (\sigma_1, \sigma_2)$ is given by

$$\begin{aligned} \hat{\underset{\sim}{\sigma}}_{ml} &= \Big(\frac{m}{r}V_x, \frac{n}{s}V_y\Big) \\ &= (\hat{\sigma}_{1ml}, \hat{\sigma}_{2ml}), \quad \text{say}, \end{aligned} \tag{2.1}$$

(see Tripathy, 2015 and Chiou and Cohen, 1984). The uniformly minimum variance unbiased estimator for the vector $\underset{\sim}{\sigma} = (\sigma_1, \sigma_2)$ is given by

$$\begin{aligned} \hat{\underset{\sim}{\sigma}}_{mv} &= \Big(\frac{m}{r}\Big(V_x + V_*^{-1}\Big), \frac{n}{s}\Big(V_y + V_*^{-1}\Big)\Big) \\ &= (\hat{\sigma}_{1mv}, \hat{\sigma}_{2mv}), \quad \text{say}, \end{aligned} \tag{2.2}$$

where $V_* = (\frac{V_x}{r-1})^{-1} + (\frac{V_y}{s-1})^{-1}$ (see Tripathy, 2015 and Chiou and Cohen, 1984).

When it is known a priori that the scale parameters follow certain ordering that is $\sigma_1 \leq \sigma_2$, these estimators need not be good enough to estimate the vector $\underset{\sim}{\sigma}$. Hence improved estimators can be obtained by using its isotonic regression with proper weights. Using the mini-max formula (see Barlow et al., 1972), one can easily write the restricted MLEs of both σ_1 and σ_2 as

$$\hat{\sigma}_{ir} = \min_{i \leq t_1 \leq k} \max_{1 \leq s_1 \leq i} Av(s_1, t_1), \quad i = 1, 2,$$

where

$$Av(s_1, t_1) = \frac{\sum_{j=s_1}^{t_1} n_j \hat{\sigma}_j}{\sum_{j=s_1}^{t_1} n_j}, s_1 \leq t_1, s_1, t_1 \in \{1, 2\}.$$

Here we denote $n_1 = r$ and $n_2 = s$. Explicitly we obtain the estimators for σ_1 and σ_2 as

$$\hat{\sigma}_{1r} = \min\Big(\frac{m}{r}V_x, \frac{mV_x + nV_y}{r+s}\Big) \quad \text{and} \quad \hat{\sigma}_{2r} = \max\Big(\frac{n}{s}V_y, \frac{mV_x + nV_y}{r+s}\Big).$$

Using these estimators for σ_1 and σ_2 we construct the restricted MLE (call it $\hat{\underset{\sim}{\sigma}}_{rm}$) of $\underset{\sim}{\sigma} = (\sigma_1, \sigma_2)$ as

$$\hat{\underset{\sim}{\sigma}}_{rm} = (\hat{\sigma}_{1r}, \hat{\sigma}_{2r}). \tag{2.3}$$

It is easy to observe that the risk of the MLE $\hat{\underset{\sim}{\sigma}}_{ml}$ and the UMVUE $\hat{\underset{\sim}{\sigma}}_{mv}$ are respectively given by

$$R(\hat{\underset{\sim}{\sigma}}_{ml}, \underset{\sim}{\sigma}) = \frac{1}{r} + \frac{1}{s},$$

and

$$R(\hat{\underset{\sim}{\sigma}}_{mv}, \underset{\sim}{\sigma}) = \frac{1}{r} + \frac{1}{s} + \Big\{\Big(\frac{m}{r\sigma_1}\Big)^2 + \Big(\frac{n}{s\sigma_2}\Big)^2\Big\}E(V_*^{-2}).$$

THEOREM 2.1 Let $\hat{\underset{\sim}{\sigma}}_{ml}$ and $\hat{\underset{\sim}{\sigma}}_{rm}$ be the MLE and the restricted MLE of $\underset{\sim}{\sigma} = (\sigma_1, \sigma_2)$:

$\sigma_1 \leq \sigma_2$ respectively. Let the loss function be the sum of the quadratic losses as given in (1.2). Then we have $R(\hat{\underset{\sim}{\sigma}}_{ml}, \underset{\sim}{\sigma}) \geq R(\hat{\underset{\sim}{\sigma}}_{rm}, \underset{\sim}{\sigma})$.

PROOF Consider the risk difference of $\hat{\underset{\sim}{\sigma}}_{rm}$ and $\hat{\underset{\sim}{\sigma}}_{ml}$:

$$\begin{aligned}
\Delta &= R(\hat{\underset{\sim}{\sigma}}_{rm}, \underset{\sim}{\sigma}) - R(\hat{\underset{\sim}{\sigma}}_{ml}, \underset{\sim}{\sigma}) \\
&= K_1 \int_1^\infty \frac{(1-z)z^{r-2}\{(1+z)-2\rho\}\{z(s-1)nr+(r-1)ms\}}{(rz+s\rho)^{r+s+1}} dz \\
&\quad + K_2 \int_1^\infty \frac{(z-1)z^{r-2}\{(1+z)-2z/\rho\}\{z(s-1)nr+(r-1)ms\}}{(rz+s\rho)^{r+s+1}} dz \\
&= \Delta_1 + \Delta_2, \quad (say),
\end{aligned}$$

where $K_1 = \frac{s^{s+1}r^{r-1}\Gamma(r+s+1)\rho^s}{(r+s)^2(m+n\rho)}$, $K_2 = \frac{s^{s-1}r^{r+1}\Gamma(r+s+1)\rho^{s+2}}{(r+s)^2(m+n\rho)}$ and $0 < \rho = \sigma_1/\sigma_2 \leq 1$. It is easy to observe that, both terms Δ_1 and Δ_2 are non-positive when $0 < \rho \leq 1$. This completes the proof of the theorem.

Next, we consider a general class of estimators for estimating the vector $\underset{\sim}{\sigma} = (\sigma_1, \sigma_2)$ and derive a sufficient condition for improving estimators in this class under the assumption, that the scale parameters are ordered, that is, $\sigma_1 \leq \sigma_2$. Consider the class of estimators

$$D_{\mathbf{c}} = \{\hat{\underset{\sim}{d}}_{\mathbf{c}} = (\hat{d}_{c_1}, \hat{d}_{c_2}) : \mathbf{c} = (c_1, c_2), c_1, c_2 \in \mathrm{R}\}, \tag{2.4}$$

where $\hat{d}_{c_1} = c_1 V_x$, and $\hat{d}_{c_2} = c_2 V_y$. This class contains the MLE with choices of $c_1 = m/r$ and $c_2 = n/s$.

To proceed further we define a vector $\mathbf{c}^*$ for the class of estimators $D_{\mathbf{c}}$ as,

$$\mathbf{c}^* = (\min(\max(c_1, c_{1*}), c_1^*), \min(\max(c_2, c_{2*}), c_2^*)), \tag{2.5}$$

where

$$c_{1*} = \frac{m(r(m+n)) - m}{mr(r-1) + nr(r+1)}, \; c_1^* = \frac{m}{r}, \; c_{2*} = \frac{n}{s+1}, \text{ and } c_2^* = \frac{n(s(m+n)) - n}{ns(s-1) + ms(s+1)}.$$

Next, we prove a general inadmissibility result for the class of estimators $D_{\mathbf{c}}$.

THEOREM 2.2 Let $\hat{\underset{\sim}{d}}_{\mathbf{c}}$ be the class of estimators for estimating the vector parameter $\underset{\sim}{\sigma}$ as given in (2.4) and the loss function be taken as in (1.2). Define a vector $\mathbf{c}^*$ as in (2.5). Then the class of estimators $\hat{\underset{\sim}{d}}_{\mathbf{c}}$ is inadmissible and is improved by $\hat{\underset{\sim}{d}}_{\mathbf{c}^*}$ if $\mathbf{c} \neq \mathbf{c}^*$.

PROOF Let us consider the risk of the estimator $\hat{\underset{\sim}{d}}_{\mathbf{c}}$ with respect to the loss function (1.2).

$$R(\hat{\underset{\sim}{d}}_{\mathbf{c}}, \hat{\underset{\sim}{\sigma}}) = E\Big(\frac{\hat{d}_{c_1} - \sigma_1}{\sigma_1}\Big)^2 + E\Big(\frac{\hat{d}_{c_2} - \sigma_2}{\sigma_2}\Big)^2.$$

The above risk is a convex function in both c_1 and c_2, hence the minimizing choices of c_1 and c_2 have been obtained as

$$\hat{c}_1 = \frac{\sigma_1 EV_x}{EV_x^2} \text{ and } \hat{c}_2 = \frac{\sigma_2 EV_y}{EV_y^2}.$$

We note that $EV_x = \frac{r}{m}\sigma_1 - p^{-1}$, $EV_y = \frac{s}{n}\sigma_2 - p^{-1}$, $EV_x^2 = \frac{n\sigma_1}{m\sigma_2}p^{-1}\{\frac{r(r-1)\sigma_2}{n} + \frac{r(r+1)\sigma_1}{m}\}$, $EV_y^2 = \frac{m\sigma_2}{n\sigma_1}p^{-1}\{\frac{s(s-1)\sigma_1}{m} + \frac{s(s+1)\sigma_2}{n}\}$ where we denote $p = \frac{m}{\sigma_1} + \frac{n}{\sigma_2}$. Substituting all these values and after some simplification we get

$$\hat{c}_1(\rho) = \frac{m(r(m+n\rho)-m)}{mr(r-1)+nr(r+1)\rho}, \quad \text{and} \quad \hat{c}_2(\rho) = \frac{n(s(m+n\rho)-n\rho)}{s(s-1)n\rho + s(s+1)m},$$

where we denote $\rho = \sigma_1/\sigma_2$; $0 < \rho \leq 1$.

In order to obtain the result we need to obtain the supremum and infimum of $\hat{c}_1(\rho)$ and $\hat{c}_2(\rho)$ with respect to ρ for fixed sample sizes. It is easy to observe that the function $\hat{c}_1(\rho)$ is a decreasing function in ρ $(0 < \rho \leq 1)$. Hence its infimum is attained as $\rho \to 1$ and supremum is attained as $\rho \to 0$. We have

$$\inf \hat{c}_1(\rho) = \frac{m(r(m+n)) - m}{mr(r-1)+nr(r+1)} = c_{1*} \text{ and } \sup \hat{c}_1(\rho) = \frac{m}{r} = c_1^*.$$

Similarly the infimum and supremum of $\hat{c}_2(\rho)$ are obtained as

$$\inf \hat{c}_2(\rho) = \frac{n}{s+1} = c_{2*} \text{ and } \sup \hat{c}_2(\rho) = \frac{n(s(m+n)) - n}{ns(s-1)+ms(s+1)} = c_2^*.$$

Utilizing these results one can easily define the vector $\mathbf{c}^*$ as given in (2.5). Now using the orbit-by-orbit improvement technique of Brewster and Zidek (1974), we have proved the theorem.

Remark 2.1 The class of estimators $D_{\mathbf{c}} = \{\hat{\underset{\sim}{d}}_{\mathbf{c}} : \mathbf{c} = (c_1, c_2),\, c_{1*} \leq c_1 \leq c_1^*,\, c_{2*} \leq c_2 \leq c_2^*\}$ is complete.

Remark 2.2 Consider the restricted parameter space $\sigma_1 \leq \sigma_2$. The estimator $\hat{\underset{\sim}{d}}_{\mathbf{c}^*}$ dominates $\hat{\underset{\sim}{d}}_{\mathbf{c}}$ if either $c_1 \in [c_{1*}, c_1^*]^c$ or $c_2 \in [c_{2*}, c_2^*]^c$. The MLE $\hat{\underset{\sim}{\sigma}}_{ml}$ can not be improved by using Theorem 2.2 as for the MLE, $c_1 \in [c_{1*}, c_1^*]$ and $c_2 \in [c_{2*}, c_2^*]$. Here $[a,b]^c$ denotes the compliment of the interval $[a,b]$ for any real numbers a and b.

In the next section, we prove some general inadmissibility results for the classes of affine and scale equivariant estimators. As a consequence estimators dominating the MLE $\hat{\underset{\sim}{\sigma}}_{ml}$, the UMVUE $\hat{\underset{\sim}{\sigma}}_{mv}$ and the restricted MLE $\hat{\underset{\sim}{\sigma}}_{rm}$ have been obtained.

3. Improving Equivariant Estimators Under Order Restrictions

In this section, we introduce the concept of invariance to our problem and derive a sufficient condition for improving estimators which are equivariant under affine group of transformations.

Let $G_A = \{g_{a,b} : g_{a,b}(x) = ax + b, a > 0, -\infty < b < \infty\}$ be a group of affine transformations. Let us define, $V_x = U_x - Z$, $V_y = U_y - Z$. Under the transformation $g_{a,b}$, the sufficient statistics being transformed as $V_x \to aV_x$, $V_y \to aV_y$ and $Z \to aZ + b$. The parameters $\mu \to a\mu + b$, and $\underset{\sim}{\sigma} \to a\underset{\sim}{\sigma}$ as $\sigma_i \to a\sigma_i, i = 1, 2$ such that the ordering remains intact. In order that the loss function (1.2) to be invariant, the estimator $\underset{\sim}{d} = (d_1, d_2)$ satisfies the relation

$$\underset{\sim}{d}(aZ + b, aV_x, aV_y) = a\underset{\sim}{d}(Z, V_x, V_y).$$

Substituting $b = -aZ$ where $a = 1/V_x$, and simplifying, we obtain the form of an affine equivariant estimator for estimating the vector prameter $\underset{\sim}{\sigma}$ based on (V_x, V_y, Z) as,

$$\begin{aligned} \underset{\sim}{d}(Z, V_x, V_y) &= V_x(\xi_1(V), \xi_2(V)), \\ &= \underset{\sim}{d}_\xi, \quad \text{(say)}, \end{aligned} \tag{3.1}$$

where $\xi = (\xi_1, \xi_2)$, $V = \frac{V_y}{V_x}$ and $\xi_i : [0, \infty) \to \mathrm{R}$, $i = 1, 2$ are real valued functions of V.

To prove the main result of this section let us define a vector valued function ξ^* for the class of estimators $\underset{\sim}{d}_\xi$ as

$$\xi^*(v) = (\min(\max(\xi_1, \xi_{1*}), \xi_1^*), \max(\xi_2, \xi_{2*})), \tag{3.2}$$

where $\xi_{1*} = m/(r + s)$, $\xi_1^* = (m + nv)/(r + s)$, and $\xi_{2*} = (m + nv)/(r + s)$.

Theorem 3.1 Let $\underset{\sim}{d}_\xi$ be the affine class of estimators as given in (3.1) for estimating the vector parameter $\underset{\sim}{\sigma}$. Let the loss function be taken as (1.2). Then the estimator $\underset{\sim}{d}_\xi$ is inadmissible and is improved by $\underset{\sim}{d}_{\xi^*}$ if there exist some values of the parameters σ_1, σ_2; $\sigma_1 \leq \sigma_2$, such that $P(\underset{\sim}{d}_\xi \neq \underset{\sim}{d}_{\xi^*}) > 0$.

Proof The proof of the theorem can be done by using a result of Brewster and Zidek (Brewster and Zidek (1974)). To complete the proof, let us consider the conditional risk function of $\underset{\sim}{d}_\xi$ given V.

$$R(\underset{\sim}{d}_\xi, \underset{\sim}{\sigma}|V = v) = E\Big\{\Big(\frac{d_{\xi_1} - \sigma_1}{\sigma_1}\Big)^2|V = v\Big\} + E\Big\{\Big(\frac{d_{\xi_2} - \sigma_2}{\sigma_2}\Big)^2|V = v\Big\}.$$

The above risk function is a convex function of both ξ_1 and ξ_2. The minimizing choices of these functions are obtained as

$$\hat{\xi}_1(v) = \frac{\sigma_1 E(V_x|V = v)}{E(V_x^2|V = v)} \quad \text{and} \quad \hat{\xi}_2(v) = \frac{\sigma_2 E(V_x|V = v)}{E(V_x^2|V = v)}. \tag{3.3}$$

It is easy to observe that, the conditional probability density function of V_x given $V = v$, is a gamma distribution with shape parameter '$r + s - 1$' and scale parameter '$\frac{\sigma_1\sigma_2}{m\sigma_2 + n\sigma_1 v}$'. Here the gamma probability density function with a shape parameter 'α' and a scale parameter 'β' is defined as

$$g(x, \alpha, \beta) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-\frac{x}{\beta}}, \quad x > 0, \alpha > 0, \beta > 0.$$

So, the conditional expectations are calculated and are obtained as

$$E(V_x|V = v) = \frac{(r + s - 1)\sigma_1\sigma_2}{m\sigma_2 + n\sigma_1 v}, \tag{3.4}$$

and

$$E(V_x^2|V=v) = (r+s-1)(r+s)\Big(\frac{\sigma_1\sigma_2}{m\sigma_2+n\sigma_1 v}\Big)^2. \tag{3.5}$$

Substituting the conditional expectations from (3.4) and (3.5) in (3.3) and simplifying we get the minimizing choices as,

$$\hat{\xi}_1(v) = \frac{m+n\rho v}{r+s} \quad \text{and} \quad \hat{\xi}_2(v) = \frac{m+n\rho v}{\rho(r+s)}.$$

In order to apply the Brewster-Zidek technique (Brewster and Zidek (1974)), it is necessary to obtain the supremum and infimum of both $\hat{\xi}_1(v)$ and $\hat{\xi}_2(v)$ for fixed values of v and fixed values of sample sizes. It is easy to note that $\hat{\xi}_1(v)$ is an increasing function of $0<\rho\le 1$ for fixed v and m,n,r,s. Hence the infimum and supremum of $\hat{\xi}_1(v)$ are obtained as

$$\inf_{0<\rho\le 1}\hat{\xi}_1(v) = \frac{m}{r+s} = \xi_{1*}, \quad \text{say,} \quad \text{and} \qquad \sup_{0<\rho\le 1}\hat{\xi}_1(v) = \frac{m+nv}{r+s} = \xi_1^*, \quad \text{say.}$$

Similarly it is easy to obtain the spremum and infimum of $\hat{\xi}_2$ and are given by

$$\inf_{0<\rho\le 1}\hat{\xi}_2(v) = \frac{m+nv}{r+s} = \xi_{2*}, \quad \text{say} \qquad \text{and} \qquad \sup_{0<\rho\le 1}\hat{\xi}_2(v) = +\infty.$$

Now using the above results it is easy to define the vector valued function ξ^* as given in (3.2). Using Theorem 3.1 of Brewster and Zidek (see Brewster and Zidek (1974)) for improving equivariant estimators we get $R(\underset{\sim}{d}_\xi,\underset{\sim}{\sigma}) \ge R(\underset{\sim}{d}_{\xi^*},\underset{\sim}{\sigma})$ when $0<\rho\le 1$. Hence the proof is completed.

Remark 3.1 The class of estimators $\underset{\sim}{d}_\xi$ such that $\xi_{1*}\le\xi_1\le\xi_1^*$ and $\xi_2\ge\xi_{2*}$ form an admissible class of estimators within the class of all estimators of the form $\underset{\sim}{d}_\xi$.

Next we apply the Theorem 3.1 to obtain improved estimators which dominate the MLE $\hat{\underset{\sim}{\sigma}}_{ml}$, the UMVUE $\hat{\underset{\sim}{\sigma}}_{mv}$ and the restricted MLE $\hat{\underset{\sim}{\sigma}}_{rm}$ when $\sigma_1\le\sigma_2$. We note that, the estimators $\hat{\underset{\sim}{\sigma}}_{ml}$, $\hat{\underset{\sim}{\sigma}}_{mv}$ and $\hat{\underset{\sim}{\sigma}}_{rm}$ belong to the class given in (3.1). Applying Theorem 3.1, we obtain the improved estimators dominating $\hat{\underset{\sim}{\sigma}}_{ml}$, $\hat{\underset{\sim}{\sigma}}_{mv}$ and $\hat{\underset{\sim}{\sigma}}_{rm}$ respectively as

$$\hat{\underset{\sim}{\sigma}}_{am} = V_x[\min(\max(\xi_{1m}(V),\xi_{1*}(V)),\xi_1^*(V)),\max(\xi_{2m}(V),\xi_{2*}(V)))], \tag{3.6}$$

where $\xi_{1m}(V)=m/r, \xi_{2m}(V)=(n/s)V$,

$$\hat{\underset{\sim}{\sigma}}_{av} = V_x[\min(\max(\xi_{1v}(V),\xi_{1*}(V)),\xi_1^*(V)),\max(\xi_{2v}(V),\xi_{2*}(V)))], \tag{3.7}$$

where $\xi_{1v}(V)=\frac{m}{r}(1+\frac{V}{(r-1)V+(s-1)}), \xi_{2v}(V)=\frac{n}{s}(V+\frac{V}{(r-1)V+(s-1)})$, and

$$\hat{\underset{\sim}{\sigma}}_{ar} = V_x[\min(\max(\xi_{1r}(V),\xi_{1*}(V)),\xi_1^*(V)),\max(\xi_{2r}(V),\xi_{2*}(V)))], \tag{3.8}$$

where

$$\xi_{1r}(V) = \begin{cases} \frac{m}{r}, & \text{if } \frac{m}{r}V_x \le \frac{n}{s}V_y, \\ \frac{m+nV}{r+s}, & \text{if } \frac{m}{r}V_x > \frac{n}{s}V_y, \end{cases}$$

$$\xi_{2r}(V) = \begin{cases} \frac{n}{s}V, & \text{if } \frac{m}{r}V_x \le \frac{n}{s}V_y, \\ \frac{m+nV}{r+s}, & \text{if } \frac{m}{r}V_x > \frac{n}{s}V_y. \end{cases}$$

REMARK 3.2 We note that the risk values of the above improved estimators could not be obtained in closed form. Hence a simulation study has been done in Section 4, to evaluate numerically the risk performances of all these estimators.

Next, we consider a smaller group of transformations which will lead to form a larger class of estimators. Consider the smaller scale group of transformations $G_S = \{g_a : g_a(x) = ax, a > 0\}$. With the help of this group structure the sufficient statistics being transformed as $Z \to aZ$, $V_x \to aV_x$ and $V_y \to aV_y$. Also the parameters $\mu \to a\mu$, $\sigma_i \to a\sigma_i$; $i = 1, 2$, so that the vector $\underset{\sim}{\sigma} \to a\underset{\sim}{\sigma}$. The loss function (1.2) will be invariant if the estimator $\underset{\sim}{\delta}$ satisfies the relation

$$\underset{\sim}{\delta}(aZ, aV_x, aV_y) = a\underset{\sim}{\delta}(Z, V_x, V_y).$$

Choosing $a = 1/V_x$, and simplifying we get the form of a scale equivariant estimator for estimating $\underset{\sim}{\sigma}$, based on (Z, V_x, V_y) as

$$\begin{aligned} \underset{\sim}{\delta}(Z, V_x, V_y) &= V_x(\psi_1(U, V), \psi_2(U, V)) \\ &= \underset{\sim}{\delta}_\psi, \text{ say} \end{aligned} \tag{3.9}$$

where $U = Z/V_x$, $V = V_y/V_x$ and ψ_1 and ψ_2 are real valued functions of U and V.

Let us define the following functions

$$\psi_1^0 = \frac{m(1+u) + n(u+v)}{r+s+1}, \ \psi_{11}^0 = \frac{m(1+u)}{r+s+1}, \psi_2^0 = \psi_1^0. \tag{3.10}$$

For the scale equivariant estimator $\underset{\sim}{\delta}_\psi$ define the vector valued function ψ^* as,

$$\psi^* = (\psi_1^*, \psi_2^*) \tag{3.11}$$

where the functions ψ_1^* and ψ_2^* are defined as

$$\psi_1^* = \begin{cases} \psi_1^0, & \text{if } u > 0, \psi_1 > \psi_1^0 \text{ or } u < 0, \psi_1 < \psi_1^0, u + v < 0, \\ \psi_{11}^0, & \text{if } u < 0, \psi_1 < \psi_{11}^0, u + v > 0, \\ \psi_1, & \text{otherwise.} \end{cases}$$

and

$$\psi_2^* = \begin{cases} \psi_2^0, & \text{if } u < 0, \psi_2 < \psi_2^0, \\ \psi_2, & \text{otherwise.} \end{cases}$$

THEOREM 3.2 Let $\underset{\sim}{\delta}_\psi$ be the class of scale equivariant estimators for estimating the vector parameter $\underset{\sim}{\sigma}$ as given in (3.9). Let the loss function be as given in (1.2). Define the vector valued function ψ^* as in (3.11). Then the estimator $\underset{\sim}{\delta}_\psi$ is inadmissible and is improved by $\underset{\sim}{\delta}_{\psi^*}$ if there exist some values of parameters $\mu, \sigma_1, \sigma_2 : \sigma_1 \le \sigma_2$, such that $P(\underset{\sim}{\delta}_{\psi^*} \ne \underset{\sim}{\delta}_\psi) > 0$.

PROOF The proof is similar to the proof of the Theorem 3.1.

Next, we use the above result to obtain estimators improving upon the MLE $\hat{\underset{\sim}{\sigma}}_{ml}$, the UMVUE $\hat{\underset{\sim}{\sigma}}_{mv}$, and the restricted MLE $\hat{\underset{\sim}{\sigma}}_{rm}$. We note that the estimators $\hat{\underset{\sim}{\sigma}}_{ml}$, $\hat{\underset{\sim}{\sigma}}_{mv}$, and

$\hat{\mathcal{Q}}_{rm}$, also belong to the class given in (3.9). As an application of Theorem 3.2, the following improved estimators have been obtained. The estimator which improves upon $\hat{\mathcal{Q}}_{ml}$ is given by

$$\hat{\mathcal{Q}}_{sm} = V_x(\psi_{1m}^*, \psi_{2m}^*) \tag{3.12}$$

where

$$\psi_{1m}^* = \begin{cases} \psi_1^0, & \text{if } u > 0, \psi_{1m} > \psi_1^0 \text{ or } u < 0, \psi_{1m} < \psi_1^0, u + v < 0, \\ \psi_{11}^0, & \text{if } u < 0, \psi_{1m} < \psi_{11}^0, u + v > 0, \\ \psi_{1m}, & \text{otherwise}, \end{cases}$$

$$\psi_{2m}^* = \begin{cases} \psi_2^0, & \text{if } u < 0, \psi_{2m} < \psi_2^0, \\ \psi_{2m}, & \text{otherwise}, \end{cases}$$

and

$$\psi_{1m} = \frac{m}{r}, \quad \psi_{2m} = \frac{n}{s} V_y.$$

The estimator which improves upon $\hat{\mathcal{Q}}_{mv}$ is given by

$$\hat{\mathcal{Q}}_{sv} = V_x(\psi_{1v}^*, \psi_{2v}^*) \tag{3.13}$$

where

$$\psi_{1v}^* = \begin{cases} \psi_1^0, & \text{if } u > 0, \psi_{1v} > \psi_1^0 \text{ or } u < 0, \psi_{1v} < \psi_1^0, u + v < 0, \\ \psi_{11}^0, & \text{if } u < 0, \psi_{1v} < \psi_{11}^0, u + v > 0, \\ \psi_{1v}, & \text{otherwise}. \end{cases}$$

$$\psi_{2v}^* = \begin{cases} \psi_2^0, & \text{if } u < 0, \psi_{2v} < \psi_2^0, \\ \psi_{2v}, & \text{otherwise}, \end{cases}$$

and

$$\psi_{1v} = \frac{m}{r}\Big(1 + \frac{V}{(r-1)V + (s-1)}\Big), \quad \psi_{2v} = \frac{n}{s}\Big(V + \frac{V}{(r-1)V + (s-1)}\Big).$$

Similarly the estimator which improves upon $\hat{\mathcal{Q}}_{rm}$ is given by

$$\hat{\mathcal{Q}}_{sr} = V_x(\psi_{1r}^*, \psi_{2r}^*) \tag{3.14}$$

where

$$\psi_{1r}^* = \begin{cases} \psi_1^0, & \text{if } u > 0, \psi_{1r} > \psi_1^0 \text{ or } u < 0, \psi_{1r} < \psi_1^0, u + v < 0, \\ \psi_{11}^0, & \text{if } u < 0, \psi_{1r} < \psi_{11}^0, u + v > 0, \\ \psi_{1r}, & \text{otherwise}. \end{cases}$$

$$\psi_{2r}^* = \begin{cases} \psi_2^0, & \text{if } u < 0, \psi_{2r} < \psi_2^0, \\ \psi_{2r}, & \text{otherwise}, \end{cases}$$

and

$$\psi_{1r} = \begin{cases} \frac{m}{r}, & \text{if } \frac{m}{r}V_x \leq \frac{n}{s}V_y, \\ \frac{m+nV}{r+s}, & \text{if } \frac{m}{r}V_x > \frac{n}{s}V_y, \end{cases}$$

$$\psi_{2r} = \begin{cases} \frac{n}{s}V, & \text{if } \frac{m}{r}V_x \leq \frac{n}{s}V_y, \\ \frac{m+nV}{r+s}, & \text{if } \frac{m}{r}V_x > \frac{n}{s}V_y. \end{cases}$$

Remark 3.3 The improved estimators $\hat{\underset{\sim}{\sigma}}_{sm}$, $\hat{\underset{\sim}{\sigma}}_{sv}$ and $\hat{\underset{\sim}{\sigma}}_{sr}$ obtained by using Theorem 3.2 have been numerically compared in Section 4.

4. Numerical Comparisons

In Section 3, we have proposed improved estimators namely $\hat{\underset{\sim}{\sigma}}_{am}$, $\hat{\underset{\sim}{\sigma}}_{av}$, $\hat{\underset{\sim}{\sigma}}_{ar}$, $\hat{\underset{\sim}{\sigma}}_{sm}$, $\hat{\underset{\sim}{\sigma}}_{sv}$, and $\hat{\underset{\sim}{\sigma}}_{sr}$, for $\underset{\sim}{\sigma} = (\sigma_1, \sigma_2)$ using Theorem 3.1 and 3.2 when there is order restriction on σ_is that is, $\sigma_1 \leq \sigma_2$. These estimators have been improved upon $\hat{\underset{\sim}{\sigma}}_{ml}$, $\hat{\underset{\sim}{\sigma}}_{mv}$, $\hat{\underset{\sim}{\sigma}}_{rm}$. In Section 2, we have also shown that the estimator $\hat{\underset{\sim}{\sigma}}_{rm}$ improves upon $\hat{\underset{\sim}{\sigma}}_{ml}$. It seems impossible to compare the risk performances of all the estimators analytically. The performance of each estimator was evaluated numerically using simulations. In order to numerically compare the performances of all the estimators, we have generated 20,000 random type-II censored samples each from two exponential populations with a common location parameter μ and different scale parameters σ_1 and σ_2 such that $\sigma_1 \leq \sigma_2$. It is easy to observe that the risk function of all these estimators with respect to the loss (1.2) is only a function of τ, where $0 < \tau = \sigma_1/\sigma_2 \leq 1$ for fixed sample sizes m, n, r and s. We note that the risk of the estimator $\hat{\underset{\sim}{\sigma}}_{ml}$ is constant $1/r + 1/s$, however the simulated risk values have been used for comparison purpose in our simulation. The percentage of relative risk for any estimator $\hat{\underset{\sim}{\sigma}}$ (say) with respect to the estimator $\hat{\underset{\sim}{\sigma}}_{mv}$ is

$$PR = \Big(1 - \frac{Risk(\hat{\underset{\sim}{\sigma}})}{Risk(\hat{\underset{\sim}{\sigma}}_{mv})}\Big) \times 100.$$

It has been observed from our simulation study that the risk values of the estimators $\hat{\underset{\sim}{\sigma}}_{rm}$, $\hat{\underset{\sim}{\sigma}}_{am}$ and $\hat{\underset{\sim}{\sigma}}_{ar}$ are very similar, hence for presentation purpose we have excluded the estimators $\hat{\underset{\sim}{\sigma}}_{ar}$ and $\hat{\underset{\sim}{\sigma}}_{am}$.

The censoring factors for the first and second populations are $k_1 = r/m$ and $k_2 = s/n$ respectively. We note that the values of k_1 and k_2 are always lie between 0 and 1. The simulation study has been done for various combinations of sample sizes and τ ranging from 0 to 1. The simulated risk values as well as the percentage of relative risk have been computed for the choices $m = n$, $m \neq n$, $k_1 = k_2$ and $k_1 \neq k_2$. The risk values of $\hat{\underset{\sim}{\sigma}}_{ml}$ (labeled as MLE) $\hat{\underset{\sim}{\sigma}}_{mv}$ (labeled as UMV), $\hat{\underset{\sim}{\sigma}}_{rm}$ (labeled as RML), $\hat{\underset{\sim}{\sigma}}_{av}$ (labeled as AMV), $\hat{\underset{\sim}{\sigma}}_{sm}$ (labeled as SML), $\hat{\underset{\sim}{\sigma}}_{sv}$ (labeled as SMV) and $\hat{\underset{\sim}{\sigma}}_{sr}$ (labeled as SRM) have been presented in the Figures 1 and 2. Specifically we have presented the risk values of all the estimators for the choices $m = n = 8$, $k_1 = k_2 = 0.25$ (Figure 1(a)), $k_1 = k_2 = 0.75$ (Figure 1(b)), $k_1 = 0.25$, $k_2 = 0.75$ (Figure 1(c)), $k_1 = 0.75$, $k_2 = 0.25$ (Figure 1(d)). The graphs for the unequal sample sizes $m = 12$, $n = 20$, $k_1 = k_2 = 0.25$ (Figure 1(e)) and $k_1 = k_2 = 0.75$ (Figure 1(f)) are also presented. Similarly, in the Figure 2(a)-2(f) the risk values have been presented for the sample sizes $m = 12, n = 20$ and $m = 20, n = 12$ with various combinations of k_1 and k_2 (mentioned in the graphs).

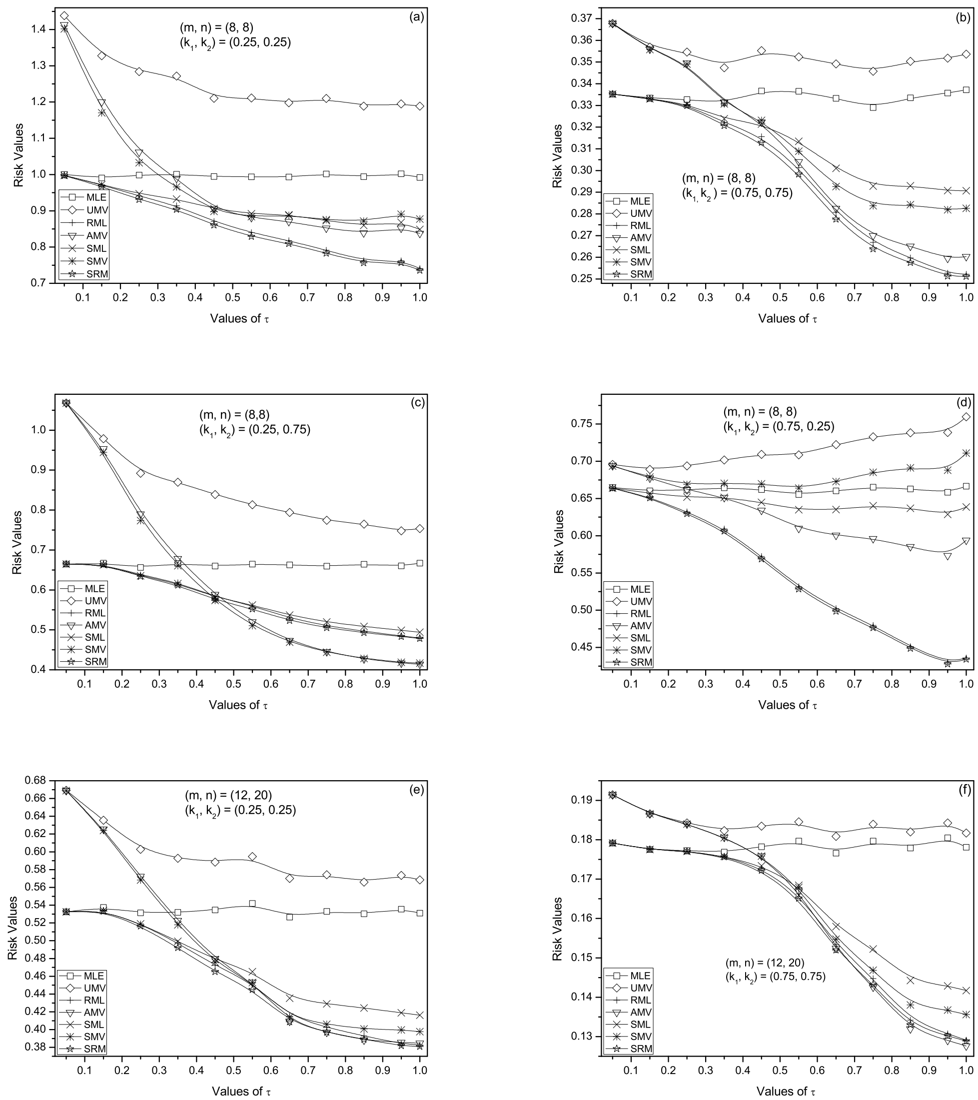

Figure 1.: Comparison of risk values of various estimators of $\underset{\sim}{\sigma}$.

The following observations have been made from our simulation study (Figure 1 and 2).

(1) The percentage of relative risk performances of each estimator with respect to $\hat{\underset{\sim}{\sigma}}_{mv}$ decreases as the censoring factors for first and second populations k_1 and k_2 increase from 0 to 1 for fixed values of m, n. However, as the sample sizes increase for fixed censoring factors (k_1 and k_2) the percentage of relative risk decreases.

(2) The percentage of relative risk improvement for $\hat{\underset{\sim}{\sigma}}_{ml}$ varies between 3% and 41%. The percentage of relative risk improvement for $\hat{\underset{\sim}{\sigma}}_{rm}$ varies between 5% and 46%. The percentage of relative risk improvement for $\hat{\underset{\sim}{\sigma}}_{av}$ varies between 0% and 51%. The percentage of relative risk improvement for $\hat{\underset{\sim}{\sigma}}_{sm}$ varies between 2% and 41%. The percentage of relative risk improvement for $\hat{\underset{\sim}{\sigma}}_{sv}$ varies between 0% and 51% whereas for $\hat{\underset{\sim}{\sigma}}_{sr}$ it is varying between 2% and 46%.

(3) The percentage of risk improvement for $\hat{\underset{\sim}{\sigma}}_{rm}$ over $\hat{\underset{\sim}{\sigma}}_{ml}$ varies between 0% and 34%. The percentage of risk improvement of $\hat{\underset{\sim}{\sigma}}_{av}$ over $\hat{\underset{\sim}{\sigma}}_{mv}$ has been quite significant and is varying between 1% and 51%. The percentage of risk improvement of $\hat{\underset{\sim}{\sigma}}_{sm}$ over

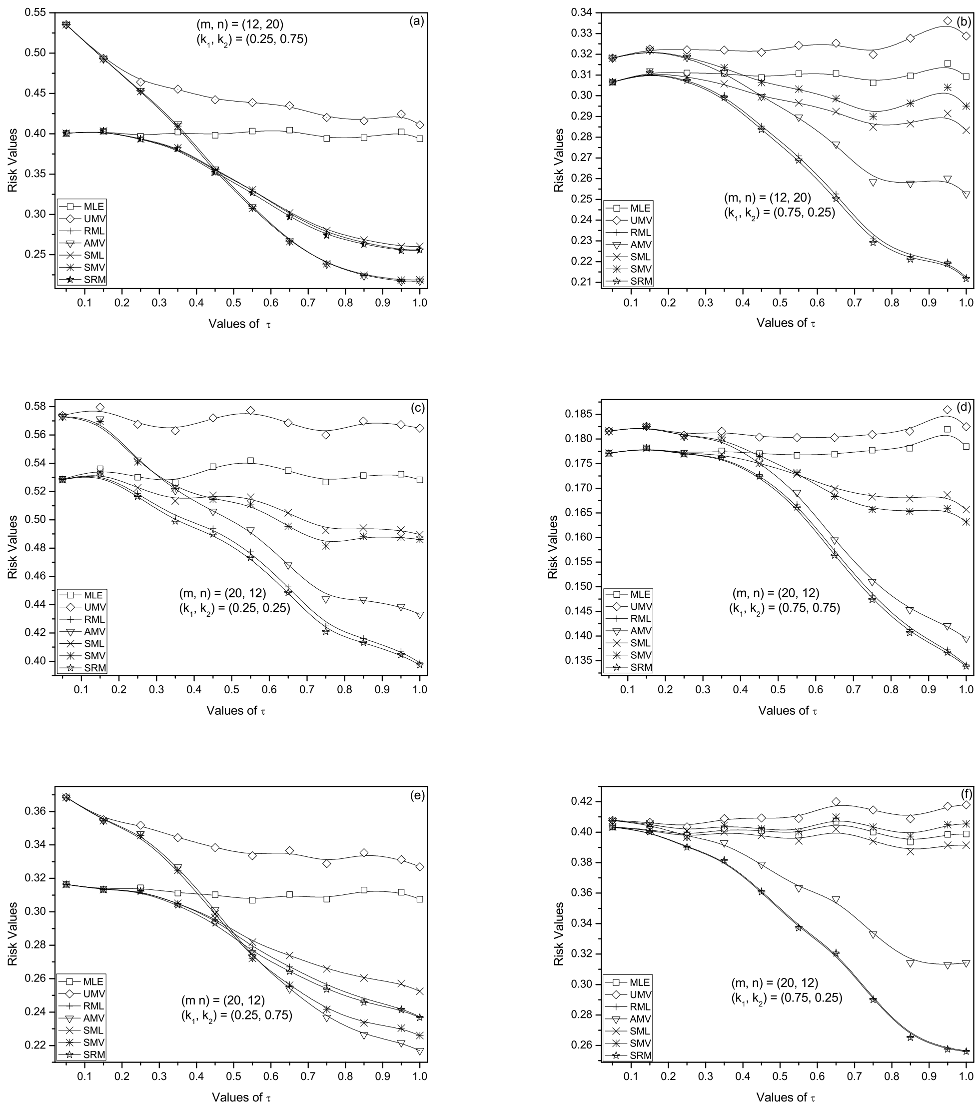

Figure 2.: Comparison of risk values of various estimators of $\underset{\sim}{\sigma}$.

$\hat{\underset{\sim}{\sigma}}_{ml}$ varies between 2% and 27%. The percentage of risk improvement for $\hat{\underset{\sim}{\sigma}}_{sv}$ over $\hat{\underset{\sim}{\sigma}}_{mv}$ varies between 0% and 51% however, for $\hat{\underset{\sim}{\sigma}}_{sr}$ over $\hat{\underset{\sim}{\sigma}}_{rm}$ is very small and is noticed between 0% and 2%.

(4) The maximum percentage of relative risk improvement has been seen for each estimator when $\tau \to 1$, and k_1 and k_2 tending to 0.
(5) For small values of τ (~0), the percentage of relative risk performance of $\hat{\underset{\sim}{\sigma}}_{sr}$ has the highest percentage of relative risk improvement ~46%. For moderate values of τ the estimator $\hat{\underset{\sim}{\sigma}}_{sr}$ also has the best performance. However, as τ approaches 1, it competes with $\hat{\underset{\sim}{\sigma}}_{av}$.
(6) Similar observations hold for other combinations of r, m and s, n.
(7) Based on above discussion and our simulation study, we recommend using the estimator $\hat{\underset{\sim}{\sigma}}_{sr}$ when the values of τ is moderate or very small (~0). For large values of $\tau \leq 1$ either of the estimators $\hat{\underset{\sim}{\sigma}}_{sr}$ or $\hat{\underset{\sim}{\sigma}}_{av}$ can be used.

Example 4.1 Suppose two brands (brand A and B) of electronic devices have been introduced in the market. It is known that the brand A uses traditional methodology where as brand B uses modern technology. The lifetimes are assumed to follow exponential distribution. It is also expected that the minimum guarantee time for both the products remain same due to market competition where as the residual life times of brand A never exceeds the residual life times of brand B. Say 20 units from each brand A and B put for a life test. Then the following failure times (in hours) from brand A and B have been observed . Brand A: 760.60, 768.34, 1159.43, 1179.04, 1224.18, 1966.99, 4125.64, 4216.05, 7554.39, 8415.60, Brand B: 259.29, 698.10, 857.57, 1471.89, 1987.32, 3486.55, 4922.22, 4941.09, 5333.26, 5869.24. Here $m = n = 20$ and $r = s = 10$. On the basis of these type-II censored samples we can easily compute $Z = 259.29$, $V_x = 5517.01$ and $V_y = 4166.65$. The various estimators for the vector parameter $\hat{\underset{\sim}{\sigma}} = (\sigma_1, \sigma_2)$ are computed as, $\hat{\underset{\sim}{\sigma}}_{ml} = (11034.04, 8333.30)$, $\hat{\underset{\sim}{\sigma}}_{mv} = (11561.56, 8860.82)$, $\hat{\underset{\sim}{\sigma}}_{rm} = (9683.67, 9683.67)$, $\hat{\underset{\sim}{\sigma}}_{av} = (9683.67, 9683.67)$, $\hat{\underset{\sim}{\sigma}}_{sm} = (9716.44, 8333.30)$, $\hat{\underset{\sim}{\sigma}}_{sv} = (9716.44, 8860.82)$, $\hat{\underset{\sim}{\sigma}}_{sr} = (9683.67, 9683.67)$. In this situation we recommend to use the estimator $\hat{\underset{\sim}{\sigma}}_{sr}$.

5. Conclusions

In this article, we have considered the simultaneous estimation of ordered scale parameters σ_is using type-II right censored samples from two exponential populations with common location parameter in a decision theoretic approach. We note that Jana and Kumar (2015) considered the componentwise estimation of ordered scale parameters when full samples (that is $r = m, s = n$) are available from two exponential populations. We have succeeded in applying Brewster and Zidek (1974) technique for simultaneous estimation of parameters. We have derived a sufficient condition for improving estimators belonging to a broad class of equivariant estimators. This class contains the MLE, and the UMVUE for estimating $\underset{\sim}{\sigma}$. As a consequence, estimators dominating the MLE, and the UMVUE in terms of risk values are obtained using the prior information $\sigma_1 \leq \sigma_2$. In fact the results obtained in this paper generalizes some of their results for simultaneous estimation of ordered scale parameters σ_is using samples from two exponential populations with a common location. Below we discuss an example where our model fits well and compute estimates for the ordered scale parameters σ_is.

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