Chilean Journal of Statistics
Vol. 6, No. 2, September 2015, 69–79

Sampling Theory
Research Paper

A general family of dual to ratio-cum-product estimators of population mean in simple random sampling

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(Received: 27 September 2013 · Accepted in final form: 04 September 2015)

Abstract

This paper presents a general family of dual to ratio-cum-product estimators of finite population mean using information on two auxiliary variates. The *bias* and *mean square error (MSE)* of the proposed family are obtained to the first degree of approximation. The expression for minimum attainable *MSE* is also derived. The proposed family encompasses a wide range of estimators of the sampling literature. Efficiency comparisons are made to demonstrate the performance of proposed family over other existing estimators. An empirical study is also carried out in support of theoretical findings.

Keywords: Auxiliary variate · Study variate · Bias · Mean square error · Dual.

Mathematics Subject Classification: Primary 62D05

1. Introduction

The problem of estimation of population parameters is a common issue in sample surveys relating to the field of agriculture, economics, medicine and population studies. The literature on survey sampling describes a great variety of techniques for utilizing information on auxiliary variates by ratio, product and regression methods of estimation to estimate the population parameters such as the mean and the variance of a variate under study. The information on auxiliary variates has an indispensable role in improving the precision of estimators of population parameters. For instance, in estimating the production of a crop, the area under cultivation can be treated as auxiliary variate.

Over the years, several estimators have been developed in *simple random sampling* to estimate the population mean using auxiliary information. Some noteworthy contributions in this direction have been made by various authors including Cochran (1940), Robson (1957), Murthy (1964), Singh (1967), Srivenkataramana (1980), Bandyopadhyaya (1980), Singh and Tailor (2005), Singh et al. (2005), Singh et al. (2011), Tailor et al. (2012), Vishwakarma et al. (2014) and many others. Moving along this direction, considerable developments have been made towards the formulation of estimators utilizing different approaches and techniques. Recently, Vishwakarma and Gangele (2014) have considered the problem of estimating the population mean in *two-phase sampling*, and Vishwakarma and

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ISSN: 0718-7912 (print)/ISSN: 0718-7920 (online)

Singh (2015) have suggested classes of separate and combined estimators for population mean under *stratified two-stage sampling*.

In this paper, we have inspected some of the existing estimators of the sampling literature and have presented a family of dual to ratio-cum-product estimators for the population mean of a study variate by utilizing information on two auxiliary variates, of which one is positively correlated with the study variate, while the other is negatively correlated. The ratio-cum-product estimator performs better than ratio as well as product estimators. Some of the existing estimators are identified as the members of the proposed family of estimators. The *asymptotic optimum estimators (AOEs)* in the proposed family have also been obtained.

We consider a finite population $U = \{U_1, U_2, ..., U_N\}$ consisting of N units. Let Y and (X, Z) be the study and auxiliary variates, respectively, taking the values Y_i and (X_i, Z_i) on the unit U_i $(i = 1, 2, ..., N)$ of the population U, where X is positively correlated with Y, while Z is negatively correlated with Y. Assuming that the population means $\bar{X}$ and $\bar{Z}$ of the auxiliary variates X and Z are known, a sample of size n (with $n < N$) is drawn from the population U using *simple random sampling without replacement (SRSWOR)* scheme to estimate the population mean $\bar{Y} = \sum_{i=1}^{N} Y_i/N$ of the study variate Y.

2. Existing Estimators

The classical ratio and product estimators for $\bar{Y}$ are given, respectively, by

$$\bar{y}_R = \bar{y}\Big(\frac{\bar{X}}{\bar{x}}\Big) \tag{1}$$

$$\bar{y}_P = \bar{y}\Big(\frac{\bar{z}}{\bar{Z}}\Big), \tag{2}$$

where $\bar{y} = \sum_{i=1}^{n} Y_i/n$, $\bar{x} = \sum_{i=1}^{n} X_i/n$ and $\bar{z} = \sum_{i=1}^{n} Z_i/n$ are the sample means of Y, X and Z, respectively.

Singh (1967) suggested a ratio-cum-product estimator for $\bar{Y}$ as

$$\bar{y}_{RP} = \bar{y}\Big(\frac{\bar{X}}{\bar{x}}\Big)\Big(\frac{\bar{z}}{\bar{Z}}\Big). \tag{3}$$

Singh and Tailor (2005) defined a ratio-cum-product estimator for $\bar{Y}$, using known correlation coefficient ρ_{XZ} between the auxiliary variates X and Z, as

$$\bar{y}_{ST} = \bar{y}\Big(\frac{\bar{X} + \rho_{XZ}}{\bar{x} + \rho_{XZ}}\Big)\Big(\frac{\bar{z} + \rho_{XZ}}{\bar{Z} + \rho_{XZ}}\Big). \tag{4}$$

Using the transformations $\bar{x}^* = (N\bar{X} - n\bar{x})/(N - n)$ and $\bar{z}^* = (N\bar{Z} - n\bar{z})/(N - n)$, Srivenkataramana (1980) and Bandyopadhyaya (1980) suggested a dual to ratio and product estimators, respectively, for $\bar{Y}$ as

$$\bar{y}_R^* = \bar{y}\Big(\frac{\bar{x}^*}{\bar{X}}\Big) \tag{5}$$

$$\bar{y}_P^* = \bar{y}\Big(\frac{\bar{Z}}{\bar{z}^*}\Big), \tag{6}$$

where $\bar{x}^* = (1 + g)\bar{X} - g\bar{x}$ and $\bar{z}^* = (1 + g)\bar{Z} - g\bar{z}$ are unbiased estimators of $\bar{X}$ and $\bar{Z}$, respectively, and $g = n/(N - n)$.

Singh et al. (2005) defined a dual to ratio-cum-product estimator for $\bar{Y}$ as

$$\bar{y}^*_{RP} = \bar{y}\Big(\frac{\bar{x}^*}{\bar{X}}\Big)\Big(\frac{\bar{Z}}{\bar{z}^*}\Big). \tag{7}$$

Singh et al. (2011) presented a family of dual to ratio-cum-product estimators for $\bar{Y}$ as

$$t = \bar{y}\Big(\frac{\bar{x}^*}{\bar{X}}\Big)^{\alpha_1}\Big(\frac{\bar{Z}}{\bar{z}^*}\Big)^{\alpha_2}. \tag{8}$$

Tailor et al. (2012) suggested a dual to Singh and Tailor (2005) estimator as

$$\bar{y}^*_{ST} = \bar{y}\Big(\frac{\bar{x}^* + \rho_{XZ}}{\bar{X} + \rho_{XZ}}\Big)\Big(\frac{\bar{Z} + \rho_{XZ}}{\bar{z}^* + \rho_{XZ}}\Big). \tag{9}$$

Vishwakarma et al. (2014) developed a class of dual to ratio-cum-product estimators for $\bar{Y}$ as

$$T = \bar{y}\Big(\frac{\bar{x}^* + \rho_{XZ}}{\bar{X} + \rho_{XZ}}\Big)^{\alpha_1}\Big(\frac{\bar{Z} + \rho_{XZ}}{\bar{z}^* + \rho_{XZ}}\Big)^{\alpha_2}. \tag{10}$$

To the first degree of approximation, the *mean square errors (MSEs)* of the estimators $\bar{y}_R$, $\bar{y}_P$, $\bar{y}_{RP}$ and $\bar{y}_{ST}$ are given, respectively, by

$$MSE(\bar{y}_R) = f_1\bar{Y}^2\Big[C_Y^2 + C_X^2(1 - 2K_{YX})\Big] \tag{11}$$

$$MSE(\bar{y}_P) = f_1\bar{Y}^2\Big[C_Y^2 + C_Z^2(1 + 2K_{YZ})\Big] \tag{12}$$

$$MSE(\bar{y}_{RP}) = f_1\bar{Y}^2\Big[C_Y^2 + C_X^2(1 - 2K_{YX} - 2K_{ZX}) + C_Z^2(1 + 2K_{YZ})\Big] \tag{13}$$

$$MSE(\bar{y}_{ST}) = f_1\bar{Y}^2\Big[C_Y^2 + \lambda_1 C_X^2(\lambda_1 - 2K_{YX} - 2\lambda_2 K_{ZX}) + \lambda_2 C_Z^2(\lambda_2 + 2K_{YZ})\Big], \tag{14}$$

where

$$f_1 = \frac{1-f}{n},\ f = \frac{n}{N},\ K_{YX} = \frac{\rho_{YX}C_Y}{C_X},\ K_{YZ} = \frac{\rho_{YZ}C_Y}{C_Z},\ K_{ZX} = \frac{\rho_{XZ}C_Z}{C_X},\ K_{XZ} = \frac{\rho_{XZ}C_X}{C_Z},$$

$$C_Y = \frac{S_Y}{\bar{Y}},\ C_X = \frac{S_X}{\bar{X}},\ C_Z = \frac{S_Z}{\bar{Z}},\ \rho_{YX} = \frac{S_{YX}}{S_Y S_X},\ \rho_{YZ} = \frac{S_{YZ}}{S_Y S_Z},\ \rho_{XZ} = \frac{S_{XZ}}{S_X S_Z},$$

$$S_Y^2 = \frac{1}{(N-1)}\sum_{i=1}^{N}(Y_i - \bar{Y})^2,\ S_X^2 = \frac{1}{(N-1)}\sum_{i=1}^{N}(X_i - \bar{X})^2,\ S_Z^2 = \frac{1}{(N-1)}\sum_{i=1}^{N}(Z_i - \bar{Z})^2,$$

$$S_{YX} = \frac{1}{(N-1)}\sum_{i=1}^{N}(Y_i - \bar{Y})(X_i - \bar{X}),\ S_{YZ} = \frac{1}{(N-1)}\sum_{i=1}^{N}(Y_i - \bar{Y})(Z_i - \bar{Z}),$$

$$S_{XZ} = \frac{1}{(N-1)}\sum_{i=1}^{N}(X_i - \bar{X})(Z_i - \bar{Z}),\ \lambda_1 = \frac{\bar{X}}{\bar{X} + \rho_{XZ}},\ \lambda_2 = \frac{\bar{Z}}{\bar{Z} + \rho_{XZ}}.$$

3. Proposed Family of Estimators

We define a family of dual to ratio-cum-product estimators for $\bar{Y}$ as

$$T_d = \bar{y}\Big(\frac{a\bar{x}^* + b}{a\bar{X} + b}\Big)^{\alpha_1}\Big(\frac{a\bar{Z} + b}{a\bar{z}^* + b}\Big)^{\alpha_2}, \tag{15}$$

where $a(\neq 0)$ and b are either real numbers or functions of some known parameters of auxiliary variates X and Z such as the correlation coefficient, coefficient of variation, etc. Also, $\alpha_i{'}s$ $(i = 1, 2)$ are unknown constants to be determined suitably such that the *MSE* of the proposed family T_d is minimized.

4. Bias and MSE of the Proposed Family

To obtain the *bias* and *MSE* of proposed family T_d , we consider

$$\bar{y} = \bar{Y}(1 + e_0),\ \bar{x} = \bar{X}(1 + e_1),\ \bar{z} = \bar{Z}(1 + e_2).$$

Then, we have

$$\left.\begin{array}{l} E(e_0) = E(e_1) = E(e_2) = 0 \\ E(e_0^2) = f_1 C_Y^2,\ E(e_1^2) = f_1 C_X^2,\ E(e_2^2) = f_1 C_Z^2 \\ E(e_0 e_1) = f_1 \rho_{YX} C_Y C_X,\ E(e_0 e_2) = f_1 \rho_{YZ} C_Y C_Z,\ E(e_1 e_2) = f_1 \rho_{XZ} C_X C_Z \end{array}\right\}. \tag{16}$$

Now, expressing (15) in terms of e's, we have

$$T_d = \bar{Y}(1 + e_0)(1 - \omega_1 g e_1)^{\alpha_1}(1 - \omega_2 g e_2)^{-\alpha_2}, \tag{17}$$

where

$$\omega_1 = \frac{a\bar{X}}{a\bar{X} + b} \ \text{ and } \ \omega_2 = \frac{a\bar{Z}}{a\bar{Z} + b}.$$

Expanding the right hand side of (17), multiplying out and retaining the terms up to second powers of e's, we obtain

$$\begin{aligned} T_d = \bar{Y}\Big[&1 + e_0 - \alpha_1\omega_1 g e_1 + \alpha_2\omega_2 g e_2 + \frac{\alpha_1(\alpha_1 - 1)}{2}\omega_1^2 g^2 e_1^2 + \frac{\alpha_2(\alpha_2 + 1)}{2}\omega_2^2 g^2 e_2^2 \\ &- \alpha_1\omega_1 g e_0 e_1 + \alpha_2\omega_2 g e_0 e_2 - \alpha_1\alpha_2\omega_1\omega_2 g^2 e_1 e_2\Big] \end{aligned}$$

or

$$\begin{aligned} T_d - \bar{Y} = \bar{Y}\Big[&e_0 - \alpha_1\omega_1 g e_1 + \alpha_2\omega_2 g e_2 + \frac{\alpha_1(\alpha_1 - 1)}{2}\omega_1^2 g^2 e_1^2 + \frac{\alpha_2(\alpha_2 + 1)}{2}\omega_2^2 g^2 e_2^2 \\ &- \alpha_1\omega_1 g e_0 e_1 + \alpha_2\omega_2 g e_0 e_2 - \alpha_1\alpha_2\omega_1\omega_2 g^2 e_1 e_2\Big]. \end{aligned} \tag{18}$$

Taking the expectation in (18) and using results of (16), we obtain the *bias* of T_d to the first degree of approximation as

$$Bias(T_d) = f_1\bar{Y}\Big[\alpha_1\omega_1 g\Big\{\frac{(\alpha_1-1)}{2}\omega_1 g C_X^2 - \rho_{YX}C_YC_X - \alpha_2\omega_2 g\rho_{XZ}C_XC_Z\Big\} + \alpha_2\omega_2 g\Big\{\frac{(\alpha_2+1)}{2}\omega_2 g C_Z^2 + \rho_{YZ}C_YC_Z\Big\}\Big].$$

Equivalently, we can write

$$Bias(T_d) = f_1\bar{Y}\Big[\alpha_1\omega_1 g C_X^2\Big\{\frac{(\alpha_1-1)}{2}\omega_1 g - K_{YX} - \alpha_2\omega_2 g K_{ZX}\Big\} + \alpha_2\omega_2 g C_Z^2\Big\{\frac{(\alpha_2+1)}{2}\omega_2 g + K_{YZ}\Big\}\Big]. \tag{19}$$

Again from (18), by neglecting the terms of e's having degree greater than one, we have

$$T_d - \bar{Y} = \bar{Y}(e_0 - \alpha_1\omega_1 g e_1 + \alpha_2\omega_2 g e_2). \tag{20}$$

Squaring both sides of (20), taking the expectation and using results of (16), we obtain the *MSE* of the proposed family T_d, to the first degree of approximation, as

$$MSE(T_d) = f_1\bar{Y}^2\Big[C_Y^2 + \alpha_1\omega_1 g(\alpha_1\omega_1 g C_X^2 - 2\rho_{YX}C_YC_X - 2\alpha_2\omega_2 g\rho_{XZ}C_XC_Z) + \alpha_2\omega_2 g(\alpha_2\omega_2 g C_Z^2 + 2\rho_{YZ}C_YC_Z)\Big].$$

Equivalently, we can write

$$MSE(T_d) = f_1\bar{Y}^2\Big[C_Y^2 + \alpha_1\omega_1 g C_X^2(\alpha_1\omega_1 g - 2K_{YX} - 2\alpha_2\omega_2 g K_{ZX}) + \alpha_2\omega_2 g C_Z^2(\alpha_2\omega_2 g + 2K_{YZ})\Big]. \tag{21}$$

4.1 Optimum Values of α_1 and α_2

As we know, α_1 and α_2 are determined such that the *MSE* of the proposed family T_d is minimized. So, the optimum values of α_1 and α_2, for which the *MSE* of T_d at (21) is minimum, are obtained on using the following condition:

$$\frac{\partial}{\partial\alpha_i}MSE(T_d) = 0\ ;\ (i = 1, 2). \tag{22}$$

On solving (22), we have

$$\alpha_{1(opt)} = \frac{(\rho_{YX} - \rho_{YZ}\rho_{XZ})C_Y}{\omega_1 g(1-\rho_{XZ}^2)C_X} = \frac{K_{YX} - K_{YZ}K_{ZX}}{\omega_1 g(1-\rho_{XZ}^2)} \tag{23}$$

$$\alpha_{2(opt)} = \frac{(\rho_{YX}\rho_{XZ} - \rho_{YZ})C_Y}{\omega_2 g(1-\rho_{XZ}^2)C_Z} = \frac{K_{YX}K_{XZ} - K_{YZ}}{\omega_2 g(1-\rho_{XZ}^2)}, \tag{24}$$

where $\alpha_{1(opt)}$ and $\alpha_{2(opt)}$ are the respective optimum values of α_1 and α_2.

Also, substitution of these optimum values of α_1 and α_2 in (21) yields the minimum attainable *MSE* of T_d as

$$MSE(T_d)_{min} = f_1\bar{Y}^2C_Y^2\Big(1 - R_{Y.XZ}^2\Big), \tag{25}$$

where $R_{Y.XZ} = \sqrt{(\rho_{YX}^2 + \rho_{YZ}^2 - 2\rho_{YX}\rho_{YZ}\rho_{XZ})\Big/(1-\rho_{XZ}^2)}$ is the multiple correlation coefficient of Y on X and Z.

Remark 4.1 The parameters C_Y, ρ_{YX} and ρ_{YZ} in (25) are generally unknown. However, these parameters can be estimated quite accurately from the preliminary data or from the repeated surveys based on sampling over several occasions. The utilization of prior information on parameters at the estimation stage has been dealt by various authors including Murthy (1967), Reddy (1978), Srivenkataramana and Tracy (1980), Singh and Singh (1984), Tracy et al. (1998), and Singh and Ruiz Espejo (2003).

5. Members of the Proposed Family of Estimators

Some of the existing estimators are identified as members of the proposed family T_d, and have been presented in Table 1. These members are obtained on assigning suitable values to the scalars α_1, α_2, a and b in (15). The expressions for the *MSEs* of these members are obtained by mere substituting the values of α_1, α_2, a and b from Table 1 in (21). It has also been verified, theoretically as well as empirically, that of all the members, the members t and T attain the *minimum variance bound (MVB)* (i.e., the minimum *MSE*) as that in (25), and hence correspond to the *asymptotic optimum estimators (AOEs)* in the proposed family T_d.

Table 1. Members of the family of estimators T_d

Author	Estimator	α_1	α_2	a	b
Usual unbiased estimator	$\bar{y}$	0	0	a	b
Srivenkataramana (1980)	$\bar{y}_R^*$	1	0	1	0
Bandyopadhyaya (1980)	$\bar{y}_P^*$	0	1	1	0
Singh et al. (2005)	$\bar{y}_{RP}^*$	1	1	1	0
Singh et al. (2011)	t	α_1	α_2	1	0
Tailor et al. (2012)	$\bar{y}_{ST}^*$	1	1	1	ρ_{XZ}
Vishwakarma et al. (2014)	T	α_1	α_2	1	ρ_{XZ}

On substituting the values of α_1, α_2, a and b from Table 1 in (21), the *MSEs* of various members of T_d, to the first degree of approximation, are obtained as

$$Var(\bar{y}) = f_1\bar{Y}^2C_Y^2 \tag{26}$$

$$MSE(\bar{y}_R^*) = f_1 \bar{Y}^2 \Big[C_Y^2 + g C_X^2 (g - 2K_{YX}) \Big] \tag{27}$$

$$MSE(\bar{y}_P^*) = f_1 \bar{Y}^2 \Big[C_Y^2 + g C_Z^2 (g + 2K_{YZ}) \Big] \tag{28}$$

$$MSE(\bar{y}_{RP}^*) = f_1 \bar{Y}^2 \Big[C_Y^2 + g C_X^2 (g - 2K_{YX} - 2gK_{ZX}) + g C_Z^2 (g + 2K_{YZ}) \Big] \tag{29}$$

$$MSE(t) = f_1 \bar{Y}^2 \Big[C_Y^2 + \alpha_1 g C_X^2 (\alpha_1 g - 2K_{YX} - 2\alpha_2 g K_{ZX}) + \alpha_2 g C_Z^2 (\alpha_2 g + 2K_{YZ}) \Big] \tag{30}$$

$$\begin{aligned} MSE(\bar{y}_{ST}^*) = f_1 \bar{Y}^2 \Big[& C_Y^2 + \lambda_1 g C_X^2 (\lambda_1 g - 2K_{YX} - 2\lambda_2 g K_{ZX}) \\ & + \lambda_2 g C_Z^2 (\lambda_2 g + 2K_{YZ}) \Big] \end{aligned} \tag{31}$$

$$\begin{aligned} MSE(T) = f_1 \bar{Y}^2 \Big[& C_Y^2 + \alpha_1 \lambda_1 g C_X^2 (\alpha_1 \lambda_1 g - 2K_{YX} - 2\alpha_2 \lambda_2 g K_{ZX}) \\ & + \alpha_2 \lambda_2 g C_Z^2 (\alpha_2 \lambda_2 g + 2K_{YZ}) \Big], \end{aligned} \tag{32}$$

where

$$\lambda_1 = \frac{\bar{X}}{\bar{X} + \rho_{XZ}} \quad \text{and} \quad \lambda_2 = \frac{\bar{Z}}{\bar{Z} + \rho_{XZ}}.$$

Also, the expressions for minimum attainable *MSEs* of t and T are given by

$$MSE(t)_{min} = f_1 \bar{Y}^2 C_Y^2 \Big(1 - R_{Y.XZ}^2 \Big) \tag{33}$$

$$MSE(T)_{min} = f_1 \bar{Y}^2 C_Y^2 \Big(1 - R_{Y.XZ}^2 \Big). \tag{34}$$

Remark 5.1 The minimum *MSEs* of t and T at (33) and (34) are same and is equal to that of the minimum *MSE* of T_d at (25). Hence, the estimators t and T correspond to the *asymptotic optimum estimators (AOEs)* in the proposed family T_d.

6. Efficiency Comparisons

For making efficiency comparisons of the proposed family T_d with the existing estimators, we have from (11) to (14), (21), and (26) to (32),

(i) $MSE(T_d) < Var(\bar{y})$ if

$$C_Z^2 < \frac{A C_X^2}{B}$$

(ii) $MSE(T_d) < MSE(\bar{y}_R)$ if

$$C_Z^2 < \frac{C_X^2 (1 - 2K_{YX} + Ag)}{Bg}$$

(iii) $MSE(T_d) < MSE(\bar{y}_P)$ if

$$C_Z^2 < \frac{A g C_X^2}{Bg - 1 - 2K_{YZ}}$$

(iv) $MSE(T_d) < MSE(\bar{y}_{RP})$ if

$$C_Z^2 < \frac{C_X^2(1 - 2K_{YX} - 2K_{ZX} + Ag)}{Bg - 1 - 2K_{YZ}}$$

(v) $MSE(T_d) < MSE(\bar{y}_{ST})$ if

$$C_Z^2 < \frac{C_X^2\{\lambda_1(\lambda_1 - 2K_{YX} - 2\lambda_2 K_{ZX}) + Ag\}}{Bg - \lambda_2(\lambda_2 + 2K_{YZ})}$$

(vi) $MSE(T_d) < MSE(\bar{y}_R^*)$ if

$$C_Z^2 < \frac{C_X^2(g - 2K_{YX} + A)}{B}$$

(vii) $MSE(T_d) < MSE(\bar{y}_P^*)$ if

$$C_Z^2 < \frac{AC_X^2}{B - g - 2K_{YZ}}$$

(viii) $MSE(T_d) < MSE(\bar{y}_{RP}^*)$ if

$$C_Z^2 < \frac{C_X^2(g - 2K_{YX} - 2gK_{ZX} + A)}{B - g - 2K_{YZ}}$$

(ix) $MSE(T_d) < MSE(t)$ if

$$C_Z^2 < \frac{C_X^2\{\alpha_1(\alpha_1 g - 2K_{YX} - 2\alpha_2 g K_{ZX}) + A\}}{B - \alpha_2(\alpha_2 g + 2K_{YZ})}$$

(x) $MSE(T_d) < MSE(\bar{y}_{ST}^*)$ if

$$C_Z^2 < \frac{C_X^2\{\lambda_1(\lambda_1 g - 2K_{YX} - 2\lambda_2 g K_{ZX}) + A\}}{B - \lambda_2(\lambda_2 g + 2K_{YZ})}$$

(xi) $MSE(T_d) < MSE(T)$ if

$$C_Z^2 < \frac{C_X^2\{\alpha_1\lambda_1(\alpha_1\lambda_1 g - 2K_{YX} - 2\alpha_2\lambda_2 g K_{ZX}) + A\}}{B - \alpha_2\lambda_2(\alpha_2\lambda_2 g + 2K_{YZ})},$$

where

$$A = \alpha_1\omega_1(2K_{YX} + 2\alpha_2\omega_2 g K_{ZX} - \alpha_1\omega_1 g) \quad \text{and} \quad B = \alpha_2\omega_2(\alpha_2\omega_2 g + 2K_{YZ}).$$

7. Empirical Study

To examine the merits of the suggested estimators of $\bar{Y}$, we have considered three natural population data sets. The description of the populations and the values of various parameters are given below:

Population I - [Source: Steel and Torrie (1960)]
Y: Log of leaf burn in sec
X: Potassium percentage
Z: Chlorine percentage
$N = 30,\ n = 6,\ \bar{Y} = 0.6860,\ \bar{X} = 4.6537,\ \bar{Z} = 0.8077,\ \rho_{YX} = 0.1794,\ \rho_{YZ} = -0.4996,$
$\rho_{XZ} = 0.4074,\ C_Y = 0.4803,\ C_X = 0.2295,\ C_Z = 0.7493$

Population II - [Source: Singh (1969)]
Y: Number of females employed
X: Number of females in service
Z: Number of educated females
$N = 61,\ n = 20,\ \bar{Y} = 7.46,\ \bar{X} = 5.31,\ \bar{Z} = 179.00,\ \rho_{YX} = 0.7737,\ \rho_{YZ} = -0.2070,$
$\rho_{XZ} = -0.0033,\ C_Y^2 = 0.5046,\ C_X^2 = 0.5737,\ C_Z^2 = 0.0633$

Population III - [Source: Johnston (1972)]
Y: Percentage of hives affected by disease
X: Mean January temperature
Z: Date of flowering of a particular summer species (number of days from January 1)
$N = 10,\ n = 4,\ \bar{Y} = 52,\ \bar{X} = 42,\ \bar{Z} = 200,\ \rho_{YX} = 0.80,\ \rho_{YZ} = -0.94,\ \rho_{XZ} = -0.73,$
$C_Y^2 = 0.0244,\ C_X^2 = 0.0170,\ C_Z^2 = 0.0021$

The *percent relative efficiencies (PREs)* are obtained for various suggested estimators of $\bar{Y}$ with respect to the usual unbiased estimator $\bar{y}$ using the formula :

$$PRE(\phi,\ \bar{y}) = \frac{Var(\bar{y})}{MSE(\phi)} \times 100,$$

where $\phi = \bar{y},\ \bar{y}_R,\ \bar{y}_P,\ \bar{y}_{RP},\ \bar{y}_{ST},\ \bar{y}_R^*,\ \bar{y}_P^*,\ \bar{y}_{RP}^*,\ t,\ \bar{y}_{ST}^*,\ T,\ T_d$.

It is observed from Table 2 that:

(1) Among the members of proposed family T_d, the *PREs* of t and T are same, and is equal to that of T_d. Hence these members are more efficient than the other members for estimating the population mean $\bar{Y}$.

(2) The *asymptotic optimum estimators (AOEs)* t and T have maximum *PREs* as compared to the other existing estimators, i.e., the ratio estimator ($\bar{y}_R$), the product estimator ($\bar{y}_P$), the ratio-cum-product estimator ($\bar{y}_{RP}$), and the Singh and Tailor (2005) estimator ($\bar{y}_{ST}$). Hence these *AOEs* perform better than the other existing estimators.

Table 2. *PREs* of different estimators of $\bar{Y}$ with respect to $\bar{y}$

Estimator	Auxiliary variates used	Population I	Population II	Population III
$\bar{y}$	-	100	100	100
$\bar{y}_R$	X	94.62	205.34	276.85
$\bar{y}_P$	Z	53.33	102.16	187.08
$\bar{y}_{RP}$	X and Z	75.50	213.54	394.86
$\bar{y}_{ST}$	X and Z	142.18	213.36	383.49
$\bar{y}_R^*$	X	102.94	214.74	238.49
$\bar{y}_P^*$	Z	131.16	104.35	149.13
$\bar{y}_{RP}^*$	X and Z	143.71	235.52	401.98
t	X and Z	**174.04**	**278.09**	**1127.72**
$\bar{y}_{ST}^*$	X and Z	131.99	235.61	405.83
T	X and Z	**174.04**	**278.09**	**1127.72**
T_d	X and Z	**174.04**	**278.09**	**1127.72**

8. Discussion and Conclusion

In the present paper, a unified approach have been developed for estimating the unknown mean $\bar{Y}$ of a study variate Y by defining a family of estimators T_d. Also, the proposed family T_d encompasses a wide range of estimators of the sampling literature, that have been listed in Table 1. The efficiencies of the members of family T_d have been compared among themselves and also with the other existing estimators. It has been established, theoretically as well as empirically, that the members t and T correspond to the *asymptotic optimum estimators (AOEs)* in the proposed family T_d, while the others do not. Moreover, in all the three populations, the *AOEs* t and T have the maximum *PREs* among the other estimators, as was expected from the results of earlier sections. Thus, the theoretical results of the previous sections have been numerically justified.

Many more estimators could also be developed for specific choices of the scalars in the proposed family of estimators so as to attain the *minimum variance bound (MVB)* similar to that of T_d, which could not be superior against T_d.

The present work is mainly concerned with the estimation of unknown mean $\bar{Y}$ under *SRSWOR* scheme. It could further be extended to *double (or two-phase) sampling* and other sampling designs.

The theoretical and empirical results exhibit the superiority of the proposed family T_d over other existing estimators. Hence, the proposed family of estimators should receive considerable attention in sample surveys dealing with estimation and inferential purposes, for instance in crop yield estimation, in household income surveys and other diverse areas of sampling.

Acknowledgements

The authors are thankful to the editor-in-chief Prof. Manuel Galea and the anonymous referees for their valuable suggestions that led to the improvement of the original article in the present form.

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