Chilean Journal of Statistics
Vol. 2, No. 1, April 2011, 115–136

Goodness-of-fit
Research Paper

On a goodness-of-fit test for censored data from a location-scale distribution with applications

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(Received: 28 December 2009 · Accepted in final form: 15 March 2011)

Abstract

In this article, we propose a goodness-of-fit test for singly Type II censored samples from a general location-scale distribution with unknown parameters. The test is a generalization to censored samples of that proposed by Michael (1983), which is based on the empirical distribution function and a variance stabilizing transformation. Acceptance regions for the probability-probability and Michael's stabilized probability plots are derived. These regions allow us the possibility of visualizing which data contribute to the decision of rejecting the null hypothesis. We consider the exponential distribution with unknown location and scale parameters as a particular case. We study the distribution of the test statistic under the null hypothesis by Monte Carlo methods. The power of the test is also estimated and compared by simulations to several distributions for the alternative hypothesis and for different sample sizes and censoring proportions. We implement the obtained results in `R` language. Finally, we illustrate the proposed results by using reliability real data sets.

Keywords: Censored data · Location-scale family · PP and SP plots · R computer language.

Mathematics Subject Classification: Primary 65C05 · Secondary 60E05

1. Introduction

An important problem arises in parametric inference when it is desired to establish from which distribution a sample comes from. Extensive research has been devoted to goodness-of-fit techniques considering this problem. A well known goodness-of-fit method is the Kolmogorov-Smirnov (KS) test. This test compares the empirical cumulative distribution function (EDF) with a completely established theoretical cumulative distribution function (CDF) under the null hypothesis (H_0). Without loss of generality, this CDF can be supposed as that from a random variable (r.v.) with uniform distribution on [0,1], which is denoted by $\mathrm{U}(0,1)$. The statistic of the KS test is denoted by D.

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ISSN: 0718-7912 (print)/ISSN: 0718-7920 (online)

http://www.soche.cl/chjs

Stephens (1974) studied and compared the power of the KS test in the case of a completely known distribution with that of the tests based on the statistics V (Kuiper), W^2 (Cramer-von Mises), A^2 (Anderson-Darling) and U^2 (Watson), called of quadratic type, and the well known chi-square (χ^2) Pearson test. The quadratic type statistics are also based on the EDF. These statistics measure the difference between the EDF and the CDF by giving weights to their squared difference. The kind of chosen weight for this difference allows to obtain the mentioned quadratic statistics; see D'Agostino and Stephens (1986).

Plots are considered as helpful tools to establish the goodness-of-fit of a certain distribution to the data. A graph that allows us to visualize the coherence between the EDF and the CDF specified in H_0 is the probability-probability (PP) plot. This plot can be associated with the KS test, but it has the disadvantage that some points can be more variable than others. Several authors proposed statistics based on probability plots to assess the goodness-of-fit of a distribution to the data. For instance, Gan and Kohler (1990) proposed goodness-of-fit tests based on PP plots for the exponential, Gumbel and normal distributions under H_0.

Michael (1983) proposed a test based on the D statistic and the arcsin transformation. This transformation stabilizes the variance of the plotted points in the PP graph associated with the KS test. Michael's graph is known as the stabilized probability (SP) plot, so that the statistic of the Michael test is denoted by D_{SP}. He studied the power of the test based on D_{SP} for testing uniformity and proved that it was more powerful than the KS test under certain alternative hypotheses (H_1).

Another situation that arises quite commonly in practice is having unknown parameters for the distribution specified under H_0 belonging to the location-scale family. Thus, these parameters must be estimated. In this case, the problem of goodness-of-fit test is more complicated, because the distribution of their test statistics depends on parameter estimators, estimation method and sample size as well as the distribution considered in H_0. However, when location and scale parameters are estimated by appropriate methods, the distribution of these goodness-of-fit statistics does not depend on the true values of the unknown parameters; see D'Agostino and Stephens (1986). In particular, Lilliefors (1967) modified the KS test for testing normality with unknown parameters; see Conover (1999, Chapter 2). Also Michael (1983) proposed a modified version of the D_{SP} statistic to consider the unknown parameters for the normal distribution under H_0.

Goodness-of-fit tests for the exponential, Gumbel and Weibull models, based on the SP plot, were developed by Kimber (1985). He used the D_{SP} statistic to test the composite hypothesis of exponentiality with unknown scale parameter and location equal to zero under the null hypothesis. Coles (1989) proposed two different goodness-of-fit tests for the two-parameter Weibull model derived from the SP plot. One of these tests is based on the D_{SP} statistic while the other is based on the correlation coefficient derived from the SP plot. Coles (1989) showed that these tests were more powerful in general than the KS test, but both proposed tests and also the KS test were not consistent in detecting the normal model. Puig and Stephens (2000) calculated percentages points of the W^2, A^2 and D statistics when the Laplace model with scale and location unknown is considered in H_0. All the mentioned authors studied and compared the power of their tests, taking different distributions in H_1 and several sample sizes; see, e.g., Gan and Kohler (1990). From these studies, it can be said that there is no test among the proposed tests based on the EDF that can be pointed out as the best overall. However, the well known Shapiro-Wilk test (1965) is known to be the most powerful for testing normality; see Tiku (1974).

In many statistical applications in survival and reliability analysis, it is not possible to obtain the complete information on survival or failure times for all the specimens or units under study. This kind of data is called censored; see, e.g., Cohen (1991), Meeker and Escobar (1998), Balakrishnan and Aggarwala (2000) and Lawless (2003). Once again, a point of interest is knowing whether a censored sample comes from a specified distribution or not. Therefore, goodness-of-fit methods with censored data are useful and necessary.

The KS test with censored data was studied by Barr and Davidson (1973) and Dufour and Maag (1978) in the case of a completely known distribution for the null hypothesis. This test for the case of normality with unknown parameters can be modified by estimating the parameters with censored data using, for example, the maximum likelihood (ML) method. However, this method does not provide analytical expressions for the estimators and iterative numerical methods are needed to find them. For this reason, the ML estimation method was discarded in the past due to its computational complexity. Thus, tests for normality with unknown parameters using censored data were based on linear estimators, as those proposed by Gupta (1952). Gupta's estimates are easy to compute and have been shown to be asymptotically efficient; see Ali and Chan (1964).

The quadratic type statistics W^2, A^2 and U^2 were modified for testing normality with unknown parameters under H_0, which were estimated for censored samples using Gupta's method; for more details, see Pettitt (1976), Pettitt and Stephens (1976), Stephens (1986) and Lawless (2003). In addition, the statistics D, W^2, A^2 and U^2 were extended to case of censored samples for some distributions of the location-scale family under the null hypothesis; see D'Agostino and Stephens (1986). Also, the Michael's test based on D_{SP} was extended for censored samples in both cases: a completely specified distribution and a normal distribution with unknown parameters. In these cases, the power of the proposed tests was high, especially in detecting some alternatives. The power of the test based in D_{SP} for testing normality under H_0 is shown to be higher than the power of the corresponding KS test; see Castro-Kuriss et al. (2009, 2010). The advantage of the test based on the D_{SP} statistic is that it will allow us to draw acceptance bands and to asses whether or not to reject the null hypothesis by identifying the points that fall outside of these regions. This points, if exists, are those that conduct the rejection of H_0.

Due to the importance of the exponential distribution in reliability, great efforts have been devoted to obtain goodness-of-fit tests for exponentiality. The described methodology has been adapted for parameters known, for only the scale parameter unknown and for both parameters of scale and location unknown, as well as for complete or censored samples. Some tests are based on the EDF, others on regression and correlation type tests, while others use special properties of the distribution, such as the fact of having a constant hazard rate function or a variation coefficient equal to one. As an example of the tests based on properties of the distribution, Balakrishnan (1983) studied the Tiku test for the exponential distribution. It was showed that this test based on spacings is more powerful than the tests considered by Dyer and Harbin (1981) for this purpose.

Kimber (1985) and Gan and Kohler (1990) proposed tests based on the D_{SP} statistic for testing exponentiality. Quadratic type tests were also extended to handle this problem. The Shapiro-Wilk test for normality has been proposed as well, but it cannot successfully detect all alternatives and is not consistent with some distributions under H_1, especially with distributions having a variation coefficient of one; see D'Agostino and Stephens (1986, p. 223) and Spinelli and Stephens (1987). These authors proposed tests for exponentiality based on the EDF as well as based on regression models with uncensored data, when origin and scale parameters are unknown. Brain and Shapiro (1983) developed two regression tests for exponentiality with censored and uncensored data. One of these tests is recommended when the alternative hypothesis has monotone hazard rate function, the other one for use when it is suspected that the alternative hypothesis has a non-monotone hazard rate function. These tests did not have good power against all the considered alternative hyphoteses. As a general conclusion from all the studies about testing exponentiality with uncensored data, we can say that it is impossible to give the best procedure against all the alternatives. Nevertheless, when only some alternatives to the exponential distribution are considered, it is possible to find powerful tests (as the Shapiro-Wilk test, for example).

We can also mention a goodness-of-fit test for exponentiality based on the Kullback-Leibler information proposed by Senoglu and Sürücü (2004), which was extended for testing normality and uniformity. Unfortunately, in the last case, it is not consistent for some distributions in H_1 studied by the authors. Also, Balakrishnan et al. (2007) proposed a goodness-of-fit test based on Kullback-Leibler information for exponentiality with progressively Type-II censored data. Balakrishnan et al. (2004) proposed goodness-of-fit tests based on spacings for progressively Type II censored data from a general location-scale distribution. This work is an extension of the tests for exponentiality introduced in Huber-Carol et al. (2002, Chapter 9, pp 89-111).

The aims of this paper are (i) to extend the Michael test for distributions in the location-scale family in H_0 and Type II censored samples and (ii) to study the particular case of the exponential distribution under the null hypothesis. We point out that the methodology presented in this article for singly right Type II censoring is also valid for singly left as well as doubly censoring (right and left). In addition, the results proposed here are valid for the log-location-scale family too, as can be seen in the next section.

The article is structured as follows. Section 2 introduces the proposed test. Section 3 shows expressions for PP and SP plots with censored data and their corresponding acceptance regions. Section 4 presents the test for exponentiality, including a comparison between the powers of the proposed test and the KS and quadratic type tests with several distributions in the alternative hypothesis, and applications of this test to real censored data sets. The quantiles of the distribution of the statistic under the null hypothesis and some useful programs to perform the test and the plots with censored data can be found in the link mentioned in Section 4. Finally, in Section 5 some conclusions are drawn.

2. Goodness-of-fit Test and Censored Data

In this section 2, we provide some preliminary aspects and introduce the proposed test.

2.1 Preliminaries

An r.v. X belongs to the location-scale family of distributions if its probability density function (PDF) is given by

$$f(x;\mu,\sigma) = \frac{1}{\sigma}\, g\left(\frac{x-\mu}{\sigma}\right), \tag{1}$$

where the functional form $g(\cdot)$ is completely specified. First, we assume that the location and scale parameters, $\mu \in \mathbb{R}$ and $\sigma > 0$, respectively, of $f(x;\mu,\sigma)$ are known and $g(\cdot)$ is the standard form of the PDF $f(x;\mu,\sigma)$. The location-scale family is a rich class of distributions that includes the normal, exponential, and Gumbel models as special cases. Considering a logarithmic transformation, we propose a test that can also be applied to the log-location-scale family, where the log-normal (LN), log-logistic and Weibull distributions are its most important members; see, e.g., Meeker and Escobar (1998).

Let $X_{(1)}, \ldots, X_{(n)}$ denote the order statistics of a sample from a completely specified location-scale distribution, with CDF $G([x-\mu]/\sigma)$. In addition, let

$$U_1 = G\left(\tfrac{X_1-\mu}{\sigma}\right), \ldots, U_n = G\left(\tfrac{X_n-\mu}{\sigma}\right) \quad \text{and} \quad U_{(j)} = G\left(\tfrac{X_{(j)}-\mu}{\sigma}\right), \quad j = 1, \ldots, n,$$

denote the corresponding ordered sample, which is an ordered $\mathrm{U}(0,1)$ sample.

The Michael statistic is defined by

$$D_{\mathrm{SP}} = \max_{1 \leq j \leq n} \left\{ \frac{2}{\pi} \left| \arcsin\left(\sqrt{\frac{j-0.5}{n}}\right) - \arcsin\left(\sqrt{U_{(j)}}\right) \right| \right\}. \quad (2)$$

If $U \sim \mathrm{U}(0,1)$, the r.v. $S = 2\arcsin(\sqrt{U})/\pi$ follows a distribution with a PDF given by $f_S(s) = \pi \sin(\pi s)/2$, with $0 < s < 1$. The order statistics, denoted by $S_{(1)}, \ldots, S_{(n)}$, associated with a sample of size n from the distribution of the r.v. S, have a constant asymptotic variance, i.e., as $n \to \infty$ and $j/n \to q$, $\mathrm{Var}[n\, S_{(j)}] \to 1/\pi^2$, which is independent of q, for all $j = 1, \ldots, n$. Due to this, Michael (1983) used the arcsin transformation to stabilize the variance of the plotted points on the probability graphs associated with the KS test. The Michael SP graph is obtained by plotting the points

$$\left[\frac{2}{\pi} \arcsin\left(\sqrt{\frac{j-0.5}{n}}\right), \frac{2}{\pi} \arcsin\left(\sqrt{u_{(j)}}\right)\right], \quad j = 1, \ldots, n.$$

Consider a censored sample with null and alternative hypotheses defined as

$$\mathrm{H}_0\colon F(x) \equiv G\left(\tfrac{x-\mu}{\sigma}\right) \quad \text{against} \quad \mathrm{H}_1\colon F(x) \neq G\left(\tfrac{x-\mu}{\sigma}\right). \quad (3)$$

When the null distribution is completely specified, i.e., both μ and σ are known, by applying the transformation $G([X_{(i)} - \mu]/\sigma)$, for $i = 1, \ldots, r$, the censored sample $U_{(1)}, \ldots, U_{(r)}$ is an ordered sample from a population with $\mathrm{U}(0,1)$ distribution. Then, the expression given in Equation (2) is valid and it can be used for any distribution in the location-scale family as long as its parameters are completely specified. Moreover, without loss of generality, we can assume the U(0,1) distribution under H_0.

Let us now consider distributions with PDF given as in Equation (1), but with unknown parameters. For testing the mentioned hypotheses in (3) with unknown location and scale parameters based on censored data, these parameters must be replaced by their estimators. However, even in the case of a true null hypothesis, the defined $U_{(i)}$, for $i = 1, \ldots, r$, are not an ordered sample from the $\mathrm{U}(0,1)$ model. Therefore, the distribution of the D_{SP} statistic used for testing the given hypotheses, obtained when we replace the parameters of the distribution by their estimates, is different from that obtained in the case of known parameters. This happens because the distribution of this statistic depends on (i) the distribution considered in H_0, (ii) its parameter estimates, (iii) the estimation method and (iv) the sample size. If the parameters are estimated by using appropriate methods, then the distribution of the D_{SP} statistic does not depend on the true values of the parameters; see D'Agostino and Stephens (1986). We propose to estimate the unknown parameters using the ML method.

2.2 Type II right censoring

In order to contrast the hypotheses given in (3) in the case of unknown parameters, let $X_{(1)} < \cdots < X_{(r)}$ be the uncensored observations of a censored right sample of size n. In this case, r is fixed and $(n - r)$ observations are greater than $X_{(r)}$ and therefore the censoring proportion is $p = r/n$. We propose the next modification for the statistic D_{SP} with censored observations and unknown parameters:

$$D^{\star}_{\mathrm{SP}} = \max_{1 \leq j \leq r} \left\{ \frac{2}{\pi} \left| \arcsin\left(\sqrt{\frac{j-0.5}{n}}\right) - \arcsin\left(\sqrt{\hat{U}_{(j)}}\right) \right| \right\}, \quad (4)$$

where $\hat{U}_{(j)} = G([X_{(j)} - \hat{\mu}]/\hat{\sigma})$, for $j = 1, \ldots, r$.

In Equation (4), we compute $\widehat{\mu}$ and $\widehat{\sigma}$ with the ML estimation method. We recall that for some distributions under H_0, the estimators of the unknown parameters can be computed directly, while in some cases the obtained equations require the use of numerical methods in order to achieve the solutions; see D'Agostino and Stephens (1986), Cohen (1991) and Castro-Kuriss et al. (2009). For the KS test, the D statistic for censored observations can also be modified using ML estimates for the unknown parameters, i.e.,

$$D^{\star} = \max_{1\leq j\leq r} \left\{ \frac{2}{\pi} \left| \frac{j-0.5}{n} - \hat{U}_{(j)} \right| \right\} + \frac{0.5}{n}, \tag{5}$$

where $\hat{U}_{(i)}$ is analogously defined as in Equation (4). We call the statistics $D^{\star}_{\mathrm{SP}}$ and $D^{\star}$ modified because they must be evaluated at the parameter estimates of μ and σ.

2.3 Computation algorithm

According to Section 2.2, the following algorithm can be proposed in order to carry out the goodness-of-fit test:

(i) Compute the ML estimates of μ and σ, say $\widehat{\mu}$ and $\widehat{\sigma}$, using appropriate expressions according to the distribution considered under the null hypothesis.
(ii) Obtain $\hat{Z}_{(j)} = [X_{(j)} - \widehat{\mu}]/\widehat{\sigma}$, for $j = 1, \ldots, r$.
(iii) Determine $\hat{U}_{(j)} = G(\hat{Z}_{(j)})$, for $j = 1, \ldots, r$.
(iv) Calculate $D^{\star}_{\mathrm{SP}}$ and $D^{\star}$, which we denote by $d^{\star}_{\mathrm{SP}}$ and $d^{\star}$, by using the value of $\hat{U}_{(j)}$ obtained in (ii) and (iii).
(v) Compare $d^{\star}_{\mathrm{SP}}$ and $d^{\star}$ with the suitable quantiles for the distribution of the statistics under H_0.
(vi) Reject H_0 at a given level of significance α if the observed value of $D^{\star}_{\mathrm{SP}}$ is greater than the $(1-\alpha)$th quantile of its distribution, which we denoted by $dsp^{\star}_{1-\alpha}$. An analogous approach must be applied for $D^{\star}$.

We note that:

(i) The statistic proposed in Equation (4) can be extended to the case of singly left or doubly type II censorship.
(ii) In the case mentioned in item (i), the corresponding tables of quantiles for the test statistic must be obtained, depending on the distribution established under H_0.
(iii) In this paper, we have analyzed the case of singly right censored samples because we intend to apply the test to problems that arise in reliability analysis where this censorship is often found.

3. SP-Plot and Acceptance Regions Using $D^{\star}_{\mathrm{SP}}$

In this section, we provide expressions for PP and SP plots with censored data and their corresponding acceptance regions.

3.1 PP and SP plots

The ordered uncensored observations from singly right censored samples (or singly left, or both singly right and left) have the same position as in the whole ordered sample. Therefore, the PP and SP plots can be obtained in this case. It is important to highlight that the corresponding quantiles must be adequately computed and that only a portion of the observations from the hypothetical distribution may be plotted. Thus, the censored observations do not appear in the proposed plots. In the general case of progressive censoring, for example, this no longer holds.

Table 1. Expressions for the indicated plots with censored data.

Plot	Abscissa	Ordinate
PP	$t_j = \dfrac{j-0.5}{n}$	$u_j = G\left(\frac{x_{(j)}-\hat{\mu}}{\hat{\sigma}}\right)$
SP	$w_j = \frac{2}{\pi}\arcsin\left(\sqrt{\frac{j-0.5}{n}}\right)$	$s_j = \frac{2}{\pi}\arcsin\left(\sqrt{G\left(\frac{x_{(j)}-\hat{\mu}}{\hat{\sigma}}\right)}\right)$

3.2 Acceptance regions

In an analogous way to the uncensored case, by using the quantiles of the distribution of the $D^\star$ and $D^\star_{\text{SP}}$ statistics with censored data, it is possible to obtain acceptance regions for the PP and SP plots. Expressions for constructing these plots with censored data are shown in Table 1, while Table 2 summarizes expressions for constructing $(1-\alpha)$ acceptance regions on probability plots for censored samples based on $D^\star$ and $D^\star_{\text{SP}}$. They can also be found in Castro-Kuriss et al. (2009) in the particular case of normality under H_0. If the r uncensored observations lie within the region constructed, then the null hypothesis cannot be rejected at α level. In all the expressions presented in Tables 1 and 2, r is the number of uncensored observations, n is the whole sample size and $j = 1, \ldots, r$.

4. Testing Exponentiality

In this section, we present a test for exponentiality and carry out the numerical part of this work. Specifically, we conduct a simulation study based on Monte Carlo methods that allows us to obtain the quantiles of the proposed test statistics and compare the power of the proposed test with those of the KS and quadratic type tests, using several distributions in H_1. In addition, we illustrate the obtained results with two examples based on real censored data sets.

4.1 Test for exponentiality with censored data

For H_0 given in (3), consider the distribution

$$G\left(\frac{x-\mu}{\sigma}\right) = 1 - \exp\left(-\frac{[x-\mu]}{\sigma}\right), \quad x \geq \mu, \quad \mu \in \mathbb{R}, \tag{6}$$

which corresponds to the exponential distribution denoted by $\text{Exp}(\mu, \sigma)$.

We are interested in the case of unknown parameters in Equation (6). Most of the proposed tests based on Equation (6) in H_0 consider σ as unknown and μ known and assumed to be equal to zero. When μ is known, this can be eliminated by using the transformation $Y_{(i)} = X_{(i)} - \mu$, for $i = 1, \ldots, n$, producing an ordered sample from the $\text{Exp}(0, \sigma)$ distribution, when H_0 is true. When μ is unknown, the substitution $Y_{(i)} = X_{(i)} - X_{(1)}$, for $i = 1, \ldots, n$, can be used to eliminate μ, where $X_{(1)}$ denotes the minimum of the sample, but it may not necessarily give the most powerful test; see Spinelli and Stephens (1987). Then, we have two different tests, one where both μ and σ are unknown and another in which μ is known and, as mentioned, can be assumed to be equal to zero.

Table 2. $100[1-\alpha]\%$ acceptance regions using quantiles $d^{\star}{}_{(1-\alpha)}$ and $d^{\star}_{\mathrm{SP}}{}_{(1-\alpha)}$.

Plot	Statistic	Lines defining acceptance regions
PP	$D^{\star}$	$\left[\max\left\{t-d^{\star}{}_{(1-\alpha)}+\frac{0.5}{n},0\right\},\quad \min\left\{t+d^{\star}{}_{(1-\alpha)}-\frac{0.5}{n},1\right\}\right]$
PP	$D^{\star}_{\mathrm{SP}}$	$\left[\max\left\{\left[\sin\left(\arcsin\left(\sqrt{t}\right)-\frac{\pi}{2}d^{\star}_{\mathrm{SP}}{}_{(1-\alpha)}\right)\right]^2,0\right\},\min\left\{\left[\sin\left(\arcsin\left(\sqrt{t}\right)+\frac{\pi}{2}d^{\star}_{\mathrm{SP}}{}_{(1-\alpha)}\right)\right]^2,1\right\}\right]$
SP	$D^{\star}$	$\left[\max\left\{\frac{2}{\pi}\arcsin\left(\sqrt{\left[\sin\left(\frac{\pi}{2}w\right)\right]^2-d^{\star}{}_{(1-\alpha)}+\frac{0.5}{n}}\right),0\right\},\min\left\{\frac{2}{\pi}\arcsin\left(\sqrt{\left[\sin\left(\frac{\pi}{2}w\right)\right]^2+d^{\star}{}_{(1-\alpha)}-\frac{0.5}{n}}\right),1\right\}\right]$
SP	$D^{\star}_{\mathrm{SP}}$	$\left[\max\left\{w-d^{\star}_{\mathrm{SP}}{}_{(1-\alpha)},0\right\},\quad \min\left\{w+d^{\star}_{\mathrm{SP}}{}_{(1-\alpha)},1\right\}\right]$

4.2 ML estimation

Let $X_{(1)}, \ldots, X_{(r)}$ be an ordered censored sample from an exponential distribution, with Type II censorship to the right from a complete sample of size n and both parameters in Equation (6) being unknown. In this case, the ML estimators of the location and scale parameters are not unbiased. Hence, unbiased estimators are derived from them and the following expressions can be obtained:

$$\hat{\sigma} = \frac{\sum_{i=1}^{r} X_i + (n-r)X_{(r)} - nX_{(1)}}{r-1} \quad \text{and} \quad \hat{\mu} = X_{(1)} - \frac{\hat{\sigma}}{n}, \tag{7}$$

where $X_{(1)}$ denotes the minimum and $X_{(r)}$ denotes the maximum of the censored sample. If the sample is complete, expressions in (7) are also true with $n = r$ and provide the best linear unbiased estimator and the uniformly minimum variance unbiased estimator for the location μ and scale σ parameters. In the case $\mu = 0$, the ML estimator of the scale parameter is obtained by

$$\hat{\sigma} = \frac{\sum_{i=1}^{r} X_i + (n-r)X_{(r)}}{r}. \tag{8}$$

The expression given in Equation (8) is valid when $n = r$ and the sample is uncensored.

4.3 Quantiles of the proposed test statistic

Consider a null hypothesis as in Subsection 4.1 and both unknown parameters estimated with expressions in (7). In this case, some quantiles for $D^\star$ and $D^\star_{\mathrm{SP}}$ from Equation (4) were obtained by means of 20,000 independent Monte Carlo samples using different censoring proportions (p) and sample sizes (n). Specifically, the quantiles were obtained for values of p from 0.2 to 1 by 0.1 and for sample sizes from 20 to 90 by 10 (of course $p = 1$ leads to the quantiles of the distributions of the $D^\star_{\mathrm{SP}}$ statistic in the case of uncensored samples). Some of the obtained results for this simulation study are displayed in Tables 3-4. We also include $D^\star$ quantiles in these tables, because in literature they are usually obtained only by estimating the scale parameter and taking the location parameter fixed at a value equal to zero; see D'Agostino and Stephens (1986). Tables for the $D^\star$ statistic considering complete samples with both parameters estimated can be obtained in Spinelli and Stephens (1987). Tables 5-7 provide quantiles for statistics W^2, A^2 and U^2 for testing exponentially with censored data. More complete tables of these quantiles can be obtained from the files `DDSPCen.PDF` and `QuadCen.PDF` available at `http://chjs.deuv.cl/files`.

4.4 Implementation

`R` language is a non-commercial and open-source software for statistical computing and graphics that can be obtained from `http://www.R-project.org`; see R Development Core Team (2009). For the particular case of the exponential distribution and for any sample size and censoring proportion, we develop `R` programs (i) to obtain quantiles of the statistics $D^\star$ and $D^\star_{\mathrm{SP}}$, (ii) to estimate the unknown parameters with the ML method, which is necessary for calculating the mentioned quantiles, and (iii) to construct the plots indicated in Section 3 and compute the corresponding p-values for the obtained tests. All of these codes can also be downloaded at `http://chjs.deuv.cl/files`. We recommend using these methods for sample sizes of at least twenty (20). We consider that this will allow the interested reader to use the proposed test of exponentiality in a wide range of problems in applied statistics.

Table 3. Quantiles of the distribution of $D^{\star}_{\mathrm{SP}}$ for an exponential distribution under H_0 for parameters estimated with the ML method and censored data for the indicated values of p, n, and $1-\alpha$.

p	n	$d^{\star}_{\mathrm{SP}(0.50)}$	$d^{\star}_{\mathrm{SP}(0.75)}$	$d^{\star}_{\mathrm{SP}(0.90)}$	$d^{\star}_{\mathrm{SP}(0.95)}$	$d^{\star}_{\mathrm{SP}(0.99)}$
0.20	20	0.0424	0.0566	0.0742	0.0841	0.0985
	30	0.0427	0.0578	0.0733	0.0827	0.1015
	40	0.0425	0.0559	0.0702	0.0795	0.0972
	50	0.0409	0.0533	0.0667	0.0757	0.0932
	60	0.0398	0.0519	0.0645	0.0731	0.0899
	70	0.0384	0.0494	0.0616	0.0695	0.0861
	80	0.0374	0.0481	0.0597	0.0672	0.0820
	90	0.0363	0.0468	0.0580	0.0650	0.0797
0.40	20	0.0610	0.0793	0.0991	0.1118	0.1386
	30	0.0574	0.0741	0.0917	0.1029	0.1273
	40	0.0539	0.0687	0.0850	0.0960	0.1188
	50	0.0506	0.0643	0.0792	0.0887	0.1097
	60	0.0484	0.0613	0.0751	0.0844	0.1030
	70	0.0460	0.0579	0.0706	0.0795	0.0973
	80	0.0444	0.0555	0.0681	0.0761	0.0924
	90	0.0427	0.0538	0.0655	0.0727	0.0888
0.70	20	0.0759	0.0961	0.1179	0.1322	0.1616
	30	0.0682	0.0856	0.1042	0.1168	0.1417
	40	0.0628	0.0780	0.0952	0.1069	0.1296
	50	0.0584	0.0723	0.0875	0.0974	0.1183
	60	0.0551	0.0681	0.0824	0.0917	0.1102
	70	0.0520	0.0641	0.0771	0.0863	0.1047
	80	0.0497	0.0613	0.0739	0.0822	0.0986
	90	0.0478	0.0590	0.0709	0.0784	0.0946
0.90	20	0.0835	0.1039	0.1260	0.1403	0.1713
	30	0.0741	0.0914	0.1099	0.1228	0.1477
	40	0.0677	0.0828	0.0999	0.1116	0.1325
	50	0.0627	0.0764	0.0915	0.1013	0.1230
	60	0.0589	0.0718	0.0858	0.0955	0.1139
	70	0.0554	0.0674	0.0807	0.0895	0.1085
	80	0.0527	0.0642	0.0767	0.0848	0.1017
	90	0.0507	0.0616	0.0733	0.0809	0.0974

Table 4. Quantiles of the distribution of $D^{\star}$ for an exponential distribution under H_0 for parameters estimated with the ML method and censored data for the indicated values of p, n, and $1-\alpha$.

p	n	$d^{\star}{}_{(0.50)}$	$d^{\star}{}_{(0.75)}$	$d^{\star}{}_{(0.90)}$	$d^{\star}{}_{(0.95)}$	$d^{\star}{}_{(0.99)}$
0.20	20	0.0678	0.0841	0.0966	0.1085	0.1254
	30	0.0580	0.0712	0.0850	0.0936	0.1099
	40	0.0512	0.0634	0.0756	0.0839	0.0987
	50	0.0468	0.0573	0.0686	0.0759	0.0900
	60	0.0431	0.0530	0.0638	0.0706	0.0838
	70	0.0400	0.0492	0.0591	0.0655	0.0780
	80	0.0378	0.0465	0.0556	0.0615	0.0733
	90	0.0359	0.0443	0.0535	0.0589	0.0692
0.40	20	0.0985	0.1203	0.1431	0.1575	0.1855
	30	0.0834	0.1019	0.1211	0.1331	0.1583
	40	0.0732	0.0895	0.1070	0.1181	0.1411
	50	0.0662	0.0807	0.0962	0.1058	0.1263
	60	0.0609	0.0744	0.0884	0.0972	0.1146
	70	0.0565	0.0687	0.0817	0.0907	0.1088
	80	0.0533	0.0648	0.0771	0.0851	0.1017
	90	0.0504	0.0615	0.0733	0.0809	0.0964
0.70	20	0.1289	0.1559	0.1845	0.2023	0.2389
	30	0.1070	0.1291	0.1530	0.1682	0.1992
	40	0.0938	0.1136	0.1344	0.1485	0.1760
	50	0.0847	0.1022	0.1207	0.1328	0.1560
	60	0.0776	0.0936	0.1108	0.1218	0.1437
	70	0.0721	0.0869	0.1030	0.1138	0.1358
	80	0.0674	0.0817	0.0970	0.1069	0.1259
	90	0.0641	0.0774	0.0915	0.1010	0.1192
0.90	20	0.1416	0.1706	0.2012	0.2210	0.2610
	30	0.1176	0.1414	0.1667	0.1837	0.2170
	40	0.1027	0.1240	0.1462	0.1599	0.1897
	50	0.0929	0.1116	0.1315	0.1441	0.1716
	60	0.0852	0.1023	0.1207	0.1328	0.1567
	70	0.0790	0.0951	0.1124	0.1236	0.1475
	80	0.0739	0.0889	0.1051	0.1158	0.1374
	90	0.0701	0.0846	0.0996	0.1094	0.1296

Table 5. Quantiles of the distribution of W^2 for an exponential distribution under H_0 for parameters estimated with the ML method and censored data for the indicated values of p, n, and $1-\alpha$.

p	n	$w^{2\star}{}_{(0.50)}$	$w^{2\star}{}_{(0.75)}$	$w^{2\star}{}_{(0.90)}$	$w^{2\star}{}_{(0.95)}$	$w^{2\star}{}_{(0.99)}$
0.20	20	0.0040	0.0061	0.0090	0.0112	0.0149
	30	0.0039	0.0065	0.0099	0.0126	0.0182
	40	0.0040	0.0067	0.0105	0.0133	0.0204
	50	0.0041	0.0068	0.0108	0.0138	0.0215
	60	0.0041	0.0070	0.0111	0.0143	0.0219
	70	0.0041	0.0070	0.0112	0.0145	0.0221
	80	0.0042	0.0070	0.0114	0.0145	0.0231
	90	0.0043	0.0072	0.0117	0.0150	0.0231
0.40	20	0.0145	0.0229	0.0352	0.0442	0.0660
	30	0.0150	0.0246	0.0378	0.0481	0.0724
	40	0.0154	0.0252	0.0393	0.0503	0.0776
	50	0.0155	0.0255	0.0393	0.0506	0.0805
	60	0.0157	0.0259	0.0405	0.0519	0.0783
	70	0.0158	0.0258	0.0399	0.0517	0.0805
	80	0.0159	0.0260	0.0410	0.0520	0.0819
	90	0.0161	0.0264	0.0414	0.0536	0.0808
0.70	20	0.0399	0.0631	0.0946	0.1193	0.1778
	30	0.0407	0.0646	0.0983	0.1236	0.1850
	40	0.0418	0.0660	0.1015	0.1306	0.1947
	50	0.0419	0.0671	0.1022	0.1309	0.1946
	60	0.0422	0.0681	0.1029	0.1315	0.1988
	70	0.0423	0.0679	0.1048	0.1335	0.2042
	80	0.0426	0.0682	0.1053	0.1343	0.2041
	90	0.0432	0.0693	0.1059	0.1355	0.2054
0.90	20	0.0590	0.0917	0.1376	0.1737	0.2589
	30	0.0602	0.0939	0.1422	0.1768	0.2758
	40	0.0617	0.0964	0.1461	0.1851	0.2719
	50	0.0621	0.0977	0.1461	0.1872	0.2864
	60	0.0631	0.0984	0.1490	0.1889	0.2873
	70	0.0628	0.0986	0.1501	0.1901	0.2919
	80	0.0631	0.0990	0.1486	0.1872	0.2888
	90	0.0640	0.1006	0.1515	0.1898	0.2955

Table 6. Quantiles of the distribution of A^2 for an exponential distribution under H_0 for parameters estimated with the ML method and censored data for the indicated values of p, n, and $1-\alpha$.

p	n	$a^{2\star}{}_{(0.50)}$	$a^{2\star}{}_{(0.75)}$	$a^{2\star}{}_{(0.90)}$	$a^{2\star}{}_{(0.95)}$	$a^{2\star}{}_{(0.99)}$
0.20	20	0.0551	0.0731	0.0997	0.1157	0.1414
	30	0.0544	0.0791	0.1135	0.1384	0.1895
	40	0.0559	0.0849	0.1243	0.1555	0.2226
	50	0.0569	0.0877	0.1315	0.1634	0.2430
	60	0.0585	0.0918	0.1394	0.1760	0.2615
	70	0.0587	0.0935	0.1429	0.1820	0.2744
	80	0.0604	0.0961	0.1481	0.1874	0.2855
	90	0.0615	0.0992	0.1540	0.1946	0.2984
0.40	20	0.1144	0.1680	0.2462	0.3026	0.4350
	30	0.1200	0.1863	0.2768	0.3477	0.5184
	40	0.1249	0.1951	0.2948	0.3725	0.5781
	50	0.1276	0.2003	0.3022	0.3897	0.5962
	60	0.1299	0.2081	0.3166	0.4029	0.5997
	70	0.1320	0.2103	0.3180	0.4108	0.6382
	80	0.1345	0.2132	0.3247	0.4198	0.6643
	90	0.1363	0.2186	0.3372	0.4295	0.6666
0.70	20	0.2298	0.3462	0.5069	0.6363	0.9453
	30	0.2384	0.3660	0.5408	0.6797	1.0080
	40	0.2465	0.3812	0.5727	0.7296	1.0878
	50	0.2506	0.3884	0.5881	0.7374	1.1181
	60	0.2556	0.3994	0.5960	0.7605	1.1185
	70	0.2574	0.4029	0.6054	0.7781	1.1805
	80	0.2605	0.4064	0.6158	0.7864	1.1883
	90	0.2656	0.4153	0.6204	0.7912	1.2287
0.90	20	0.3288	0.4898	0.7170	0.8874	1.3385
	30	0.3399	0.5130	0.7533	0.9420	1.4270
	40	0.3546	0.5344	0.7922	0.9870	1.4663
	50	0.3593	0.5467	0.8052	1.0055	1.5244
	60	0.3672	0.5564	0.8238	1.0316	1.5392
	70	0.3673	0.5573	0.8281	1.0479	1.5885
	80	0.3710	0.5638	0.8356	1.0410	1.5802
	90	0.3779	0.5752	0.8484	1.0541	1.6074

Table 7. Quantiles of the distribution of U^2 for an exponential distribution under H_0 for parameters estimated with the ML method and censored data for the indicated values of p, n, and $1-\alpha$.

p	n	$u^{2\star}{}_{(0.50)}$	$u^{2\star}{}_{(0.75)}$	$u^{2\star}{}_{(0.90)}$	$u^{2\star}{}_{(0.95)}$	$u^{2\star}{}_{(0.99)}$
0.20	20	0.0019	0.0027	0.0038	0.0046	0.0061
	30	0.0022	0.0032	0.0046	0.0055	0.0077
	40	0.0023	0.0035	0.0050	0.0061	0.0085
	50	0.0024	0.0036	0.0052	0.0063	0.0089
	60	0.0025	0.0038	0.0054	0.0066	0.0091
	70	0.0025	0.0038	0.0055	0.0067	0.0093
	80	0.0026	0.0039	0.0055	0.0067	0.0096
	90	0.0026	0.0040	0.0057	0.0070	0.0098
0.40	20	0.0088	0.0131	0.0187	0.0230	0.0327
	30	0.0096	0.0144	0.0208	0.0255	0.0366
	40	0.0099	0.0151	0.0216	0.0267	0.0379
	50	0.0102	0.0154	0.0221	0.0272	0.0398
	60	0.0103	0.0156	0.0222	0.0275	0.0386
	70	0.0104	0.0157	0.0226	0.0282	0.0410
	80	0.0106	0.0158	0.0227	0.0281	0.0410
	90	0.0106	0.0162	0.0235	0.0287	0.0411
0.70	20	0.0280	0.0429	0.0612	0.0762	0.1124
	30	0.0295	0.0450	0.0649	0.0798	0.1152
	40	0.0306	0.0464	0.0675	0.0837	0.1212
	50	0.0311	0.0472	0.0677	0.0842	0.1224
	60	0.0315	0.0480	0.0694	0.0848	0.1224
	70	0.0317	0.0480	0.0700	0.0870	0.1275
	80	0.0318	0.0484	0.0708	0.0880	0.1279
	90	0.0326	0.0488	0.0713	0.0883	0.1283
0.90	20	0.0456	0.0682	0.0983	0.1202	0.1734
	30	0.0475	0.0710	0.1020	0.1259	0.1805
	40	0.0486	0.0733	0.1056	0.1290	0.1875
	50	0.0493	0.0740	0.1067	0.1308	0.1903
	60	0.0499	0.0747	0.1078	0.1329	0.1930
	70	0.0499	0.0751	0.1090	0.1351	0.1967
	80	0.0502	0.0753	0.1093	0.1359	0.1969
	90	0.0513	0.0763	0.1101	0.1356	0.1988

4.5 Illustrative examples

Here, for the purposes of illustration, we apply the new goodness-of-fit test to real problems that arise in the field of reliability.

4.5.1 Example 1

A study conducted by Dr. William Meeker in 1999 consisted of a factorial experiment to compare the lifetimes of springs as a function of a processing temperature and of the amount of displacement in the spring test (stroke). The r.v. of interest corresponds to k-cycles obtained with the stroke at a temperature level with two different methods, one of them was a new method while the other was the old one. In each temperature level, the status for each observation (failed or not, i.e., if the observation is complete or censored, respectively) was registered. The sample consists of $n = 36$ observations with three of them censored at 5000 k-cycles. This type II censored sample is decreasingly ordered with respect to the r.v. k-cycles at a temperature level of 500, regardless of the employed method. The data set was provided to the author by Dr. Luis A. Escobar and it is summarized in Table 8, where the indicated values with $\star$ are the censored observations.

We want to test whether this censored sample can come from an exponential distribution with both unknown parameters or not. The parameters were estimated to be $\hat{\mu} = 144.54$ and $\hat{\sigma} = 2392.69$ using expressions in (7). We consider the proposed tests based on $D^\star$ and $D^\star_{\mathrm{SP}}$ for H_0 and obtain the observed statistics $d^\star = 0.1165\,(0.3 < \text{p-value} < 0.4)$ and $d^\star_{\mathrm{sp}} = 0.0655\,(0.5 < \text{p-value} < 0.6)$. Both tests do not reject the hypothesis of exponentiality under H_0. We construct the PP and SP plots with the mentioned `R` code and, as expected, all the observations fall inside of the 95% acceptance region of $D^\star_{\mathrm{sp}}$, as well as the 95% region of $D^\star$, as can be seen in the PP and SP plots of Figure 1. The values of the observed statistics $d^\star$ and $d^\star_{\mathrm{sp}}$ are obtained in the 16th ordered observation that corresponds to 1967 k-cyles.

Table 8. Data set New Spring.

K-cycles	Temp.	Method	K-cycles	Temp.	Method	K-cycles	Temp.	Method
211	500	Old	1065	500	Old	3199	500	New
218	500	Old	1563	500	Old	3464	500	New
319	500	Old	1756	500	Old	3644	500	Old
551	500	Old	1967	500	Old	3674	500	Old
638	500	Old	1995	500	Old	3904	500	Old
650	500	Old	2029	500	Old	4006	500	Old
707	500	Old	2193	500	Old	4196	500	New
712	500	Old	2287	500	New	4542	500	New
752	500	New	2592	500	New	5000	500	New
834	500	Old	2785	500	Old	5000⋆	500	New
997	500	Old	2843	500	New	5000⋆	500	New
1016	500	New	2853	500	New	5000⋆	500	Old

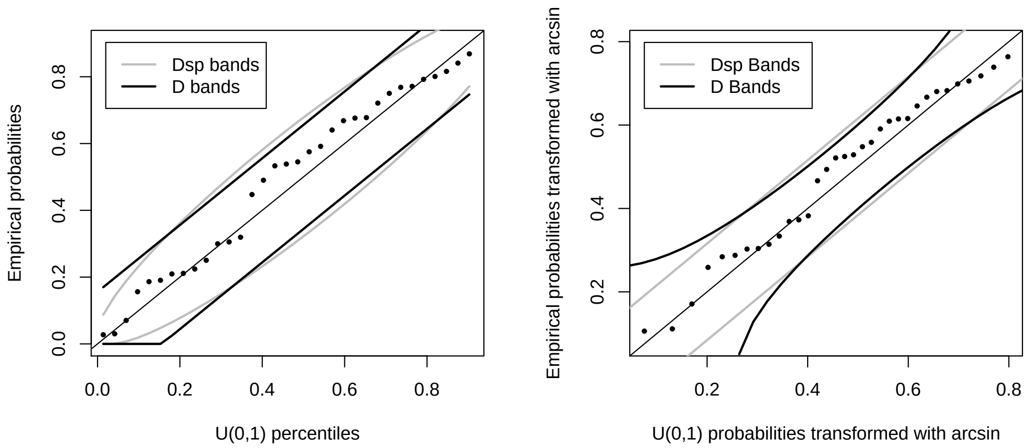


Figure 1. PP (left) and SP plots of the r.v. k-cyles at temp. 500 of New Spring Data

4.5.2 Example 2

Lawless (2003) analyzed an example with data presented in Wilk et al. (1962). These data consisted of lifetimes of transistors obtained from an accelerated life test. The lifetimes are singly Type II censored and come from a sample of size $n = 34$, with three censored observations. The lifetimes (in weeks) are given in Table 9 (three of them are censored and denoted by asterisks). As can be seen, the data are heavily rounded off.

Table 9. Wilk data.

3	4	5	6	6	7	8	8	9	9	9	10	10	11	11	11	13
13	13	13	13	17	17	19	19	25	29	33	42	42	52	52⋆	52⋆	52⋆

We want to test whether the censored sample could come from a two-parameter exponential distribution. In this case, the parameters were estimated to be $\hat{\mu} = 2.4696$ and $\hat{\sigma} = 18.0333$. We consider both proposed tests as in Example 1 and obtain the following values for the observed statistics, $d^\star = 0.1753\,(0.01 < \text{p-value} < 0.05)$ and $d^\star_{\rm sp} = 0.1028\,(0.1 < \text{p-value} < 0.15)$. At a 5% level, the test based on $D^\star$ rejects H_0 and there is one observation out of the corresponding 95% acceptance region, while all the observations fall inside of the 95% region of $D^\star_{\rm sp}$, as can be seen in Figures 2. In these figures, the observation falling outside of these regions is indicated. The plots also show an asymmetric distribution and that the plotted points do not follow a straight line. This indicates a bad specification of the postulated hypothetical distribution, in this case, the exponential model. Our results are consistent with those obtained by Lawless (2003).

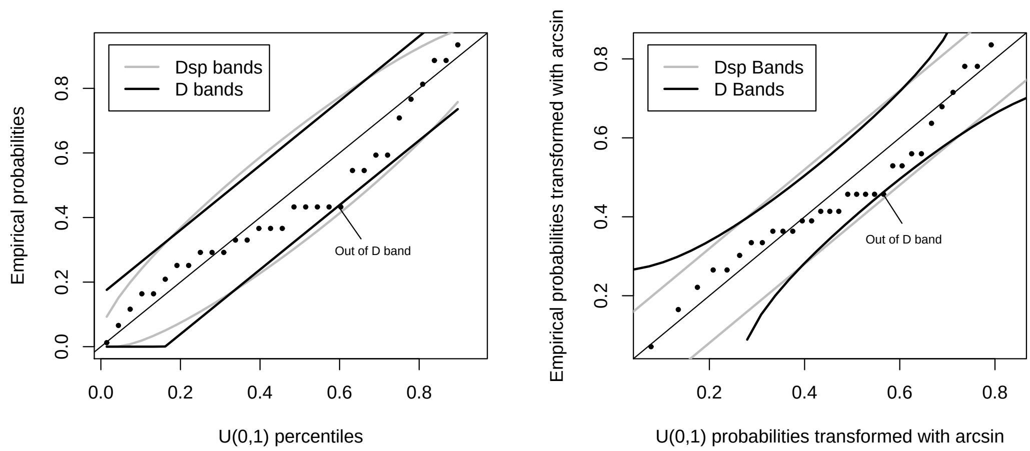


Figure 2. PP (left) and SP plots of the Wilk data

4.6 Monte Carlo power comparison

As mentioned, in order to compare the power of the proposed test, a Monte Carlo study was conducted. In this study, 20,000 independent samples were generated for different sample sizes and censoring proportions. Censored samples with $p = 0.3, 0.6, 0.8$ were considered. We have taken both parameters to be unknown and they were estimated by using the ML estimation method by expressions in (7). The level of significance was established at $\alpha = 0.05$. The obtained results for this simulation study are summarized in Tables 10-15 and have been graphically displayed in Figures 4-3. We consider distributions having different properties: with increasing failure rate (as the Weibull distribution with parameter greater than one, location equal to zero and scale equal to one), decreasing failure rate (as the Weibull distribution with parameter less than one, location equal to zero and scale equal to one), non-monotone failure rate (as the LN distribution), short tails (as the uniform distribution), heavy tails (as the Cauchy distribution), symmetric (as the Student-t distribution), and skew distributions (as the chi-squared distribution). Some of these distributions were previously considered by other authors for analyzing exponentiality under H_0. We consider eighteen (18) distributions under H_1, which are (i) chi-squared with three different parameters, (ii) U(0,1), (iii) LN, (iv) half-Cauchy (HCau), (v) half-normal (HN), (vi) Weibull with four different parameters (but scale equal to one and location equal to zero), (vii) beta(2, 1), (viii) one-parameter Lomax, (ix) N(0, 1), (x) Cauchy, (xi) Student-t and (xii) Laplace distributions. All the tests achieved the nominal level, which was verified by taking 20,000 exponential samples in the simulation study. We compare the proposed test power with that of the quadratic type tests for the null hypothesis considered, with parameters also estimated with the ML method. It was also necessary to obtain the corresponding quantiles of the distribution under H_0 of each of the statistics defined for those tests. Altogether, the comparative power of five tests was analyzed.

From Tables 10-15 and from Figures 4-3, we have the following observations:

(i) As expected, for the analyzed tests and for every distribution considered in H_1, the power increases as the sample size increases. When the proportion of uncensored observations increases, the power increases too for every distribution.

(ii) Under H_0, the empirical power, as expected, is close to the nominal level. This can be seen in the last row of the panel corresponding to each value of p in Tables 10-15.

(iii) The power of all the tests is low when p is small.

(iv) There is no test that can be pointed out as the most powerful overall for the alternatives, sample sizes and censoring proportions considered. Among the five considered tests, that based on the U^2 statistics was in general out-performed by the others.

(v) All the tests have little power for detecting the HN, HCau, Lomax with parameter equal to two and LN distributions.
(vi) In the particular case of the LN distribution, the test based on $D^{\star}_{\mathrm{SP}}$ is better than the others for all the sample sizes and proportion censoring considered.
(vii) The test based on $D^{\star}_{\mathrm{SP}}$ is in general more powerful than that based on $D^{\star}$, while within the quadratic type tests that based on W^2 is in general more powerful, especially when n is small. With sample sizes over fifty (50), the power of the test based on A^2 is similar and sometimes slightly better than the power of the test based on W^2.
(viii) The test based on $D^{\star}_{\mathrm{SP}}$ is more powerful for the increasing failure rate distributions or, at least, is a good competitor of the better one.
(ix) All the tests have good power in detecting the normal, Cauchy, Student-T and Laplace distributions. Due to this reason, we have not included figures of the estimated power of the studied tests with these distributions as alternatives.

5. Concluding Remarks

In this paper, we have proposed a new goodness-of-fit test for a location-scale distribution under the null hypothesis. It was most comprehensively studied in the case of the exponential family and it turns out to be powerful for most of the alternatives considered, particularly when the proportion of complete observations is high. Among all the tests previously proposed and studied for testing exponentiality by other authors including that presented here, it is not possible to obtain one powerful for all the alternatives considered. We also notice that a test can have good power with complete samples while, extending it to censored samples it can lose its power and can even become inconsistent for some alternatives. We understand that the advantage of the tests based on the statistics D and D_{SP} is the possibility of drawings plots with acceptance regions, where not only the rejection or not of the null hypothesis can be observed but also the points that make this decision. This possibility is only obtained with these statistics of KS type based on the EDF, while other tests designed for equivalent situations, such as the quadratic type tests, or the goodness-of-fit tests based on Kullback-Leibler information, have acceptance regions that cannot be drawn in probability plots. We are also developing a more extensive `R` package to perform the estimations, tests and plots that will allow practitioners to use the proposed tests in practical problems that will include not only the case of exponentiality but other distributions under H_0 as the normal and log-normal distributions.

Acknowledgments

The author wishes to thank Dr. Luis A. Escobar and Dr. William Meeker for providing the database New Spring and also the Executive Editor, Dr. Víctor Leiva, the Associate Editor, and the two referees for their helpful comments. This paper was partially completed while the author was carrying out research at the Department of Statistics, Universidad de Valparaíso, Chile. She is very grateful for the kind hospitality she found in the colleagues and staff.

Table 10. Estimated power (in %) of tests based on the indicated statistics, models and values of n for $p = 0.3$ and $\alpha = 0.05$.

	$n=20$					$n=30$					$n=40$				
Model	$D^\star$	$D^\star_{\text{sp}}$	W^2	A^2	U^2	$D^\star$	$D^\star_{\text{sp}}$	W^2	A^2	U^2	$D^\star$	$D^\star_{\text{sp}}$	W^2	A^2	U^2
Bet(2,1)	16	16	17	15	5	24	24	26	24	8	31	32	36	33	12
HCa(0,1)	5	5	5	5	5	5	6	5	5	5	5	5	5	5	4
HN(0,1)	6	6	6	6	4	6	6	6	6	5	6	6	7	6	4
LN(0,1)	8	8	8	7	4	9	9	9	8	4	10	12	11	10	5
Lom(0.5)	4	4	4	4	6	4	4	4	5	6	4	4	4	4	5
Lom(2)	4	4	3	5	8	5	4	4	6	9	5	4	5	7	10
Wei(0.5)	7	8	6	12	21	15	17	16	25	31	25	27	27	39	39
Wei(2)	12	12	13	12	4	18	19	20	18	6	23	25	27	24	9
Wei(3)	17	18	19	17	6	28	28	31	29	10	38	39	43	41	17
Wei(6)	24	24	27	24	8	40	39	44	41	18	54	54	60	57	30
$\chi^2(1)$	6	6	5	10	17	11	12	12	19	24	18	20	19	30	30
$\chi^2(3)$	7	8	8	7	4	9	9	9	8	4	10	11	11	9	4
$\chi^2(4)$	9	10	10	9	4	12	13	13	12	5	16	17	17	16	6
Cau(0,1)	93	92	94	93	79	99	99	99	99	96	100	100	100	100	99
Lap(0,1)	76	75	81	79	44	93	92	96	95	77	99	98	99	99	92
N(0,1)	52	54	58	55	22	76	75	81	79	49	88	87	92	91	68
$t(3)$	74	74	78	76	47	92	92	94	94	77	98	97	99	98	91
U(0,1)	6	6	7	6	4	7	7	7	6	5	7	7	7	6	4
Exp(0,1)	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5

Table 11. Estimated power (in %) of tests based on the indicated statistics, models and values of n for $p = 0.3$ and $\alpha = 0.05$.

	$n=50$					$n=70$					$n=90$				
Model	$D^\star$	$D^\star_{\text{sp}}$	W^2	A^2	U^2	$D^\star$	$D^\star_{\text{sp}}$	W^2	A^2	U^2	$D^\star$	$D^\star_{\text{sp}}$	W^2	A^2	U^2
Bet(2,1)	39	40	45	42	17	56	55	64	61	29	69	68	77	75	40
HCa(0,1)	6	6	6	5	5	6	6	6	6	5	6	6	6	6	4
HN(0,1)	7	7	7	6	5	7	7	8	7	5	8	7	8	7	5
LN(0,1)	13	16	14	13	6	17	22	20	20	8	21	29	25	26	10
Lom(0.5)	5	4	5	5	6	5	4	5	5	6	5	4	5	5	6
Lom(2)	7	5	7	9	11	9	6	10	12	12	11	8	12	15	13
Wei(0.5)	35	38	39	52	47	55	59	60	73	62	68	73	74	85	71
Wei(2)	31	33	35	33	13	44	46	51	49	21	56	59	65	63	30
Wei(3)	50	51	56	53	26	68	68	76	73	42	81	81	87	86	57
Wei(6)	68	67	74	71	43	85	83	90	88	64	93	92	96	95	79
$\chi^2(1)$	26	29	29	41	37	41	45	45	60	49	54	60	60	74	58
$\chi^2(3)$	12	14	13	11	5	16	18	18	17	7	18	22	22	21	8
$\chi^2(4)$	18	21	21	20	7	27	31	32	31	12	35	40	42	42	17
Cau(0,1)	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100
Lap(0,1)	100	100	100	100	98	100	100	100	100	100	100	100	100	100	100
N(0,1)	95	95	97	97	83	99	99	100	100	96	100	100	100	100	99
$t(3)$	99	99	100	100	97	100	100	100	100	100	100	100	100	100	100
U(0,1)	8	7	8	7	5	8	8	9	8	5	9	9	10	9	5
Exp(0,1)	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5

Table 12. Estimated power (in %) of tests based on the indicated statistics, models and values of n for $p = 0.6$ and $\alpha = 0.05$.

	$n=20$					$n=30$					$n=40$				
Model	$D^\star$	$D^\star_{\rm sp}$	W^2	A^2	U^2	$D^\star$	$D^\star_{\rm sp}$	W^2	A^2	U^2	$D^\star$	$D^\star_{\rm sp}$	W^2	A^2	U^2
Bet(2,1)	42	42	48	45	13	65	63	72	69	30	79	76	85	83	47
HCa(0,1)	5	5	5	5	6	6	5	6	6	7	5	5	6	5	6
HN(0,1)	7	8	8	7	3	9	10	10	9	5	10	11	11	10	5
LN(0,1)	6	7	6	6	4	8	10	8	8	5	8	13	9	9	6
Lom(0.5)	24	22	28	31	34	39	34	44	48	47	49	43	55	58	55
Lom(2)	5	5	6	7	8	7	5	7	8	9	7	5	8	8	10
U(0,1)	12	12	12	11	4	16	15	17	15	6	18	17	21	17	7
Wei(0.5)	33	36	40	48	48	56	60	63	72	66	71	74	77	84	78
Wei(2)	24	26	27	25	7	40	42	45	44	17	53	55	60	58	27
Wei(3)	40	42	46	43	14	63	64	70	68	33	79	78	85	83	52
Wei(6)	57	58	64	61	24	81	79	86	84	53	92	91	95	94	74
$\chi^2(1)$	19	21	23	30	31	34	36	39	49	44	45	49	51	62	55
$\chi^2(3)$	9	10	9	8	3	11	14	12	12	5	14	17	15	14	7
$\chi^2(4)$	13	15	14	13	4	21	24	23	22	8	26	31	30	29	12
Cau(0,1)	93	92	94	93	79	99	99	99	99	96	100	100	100	100	99
Lap(0,1)	76	75	81	79	44	93	92	96	95	77	99	98	99	99	92
N(0,1)	52	54	58	55	22	76	75	81	79	49	88	87	92	91	68
$t(3)$	74	74	78	76	47	92	92	94	94	77	98	97	99	98	91
Exp(0,1)	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5

Table 13. Estimated power (in %) of tests based on the indicated statistics, models and values of n for $p = 0.6$ and $\alpha = 0.05$.

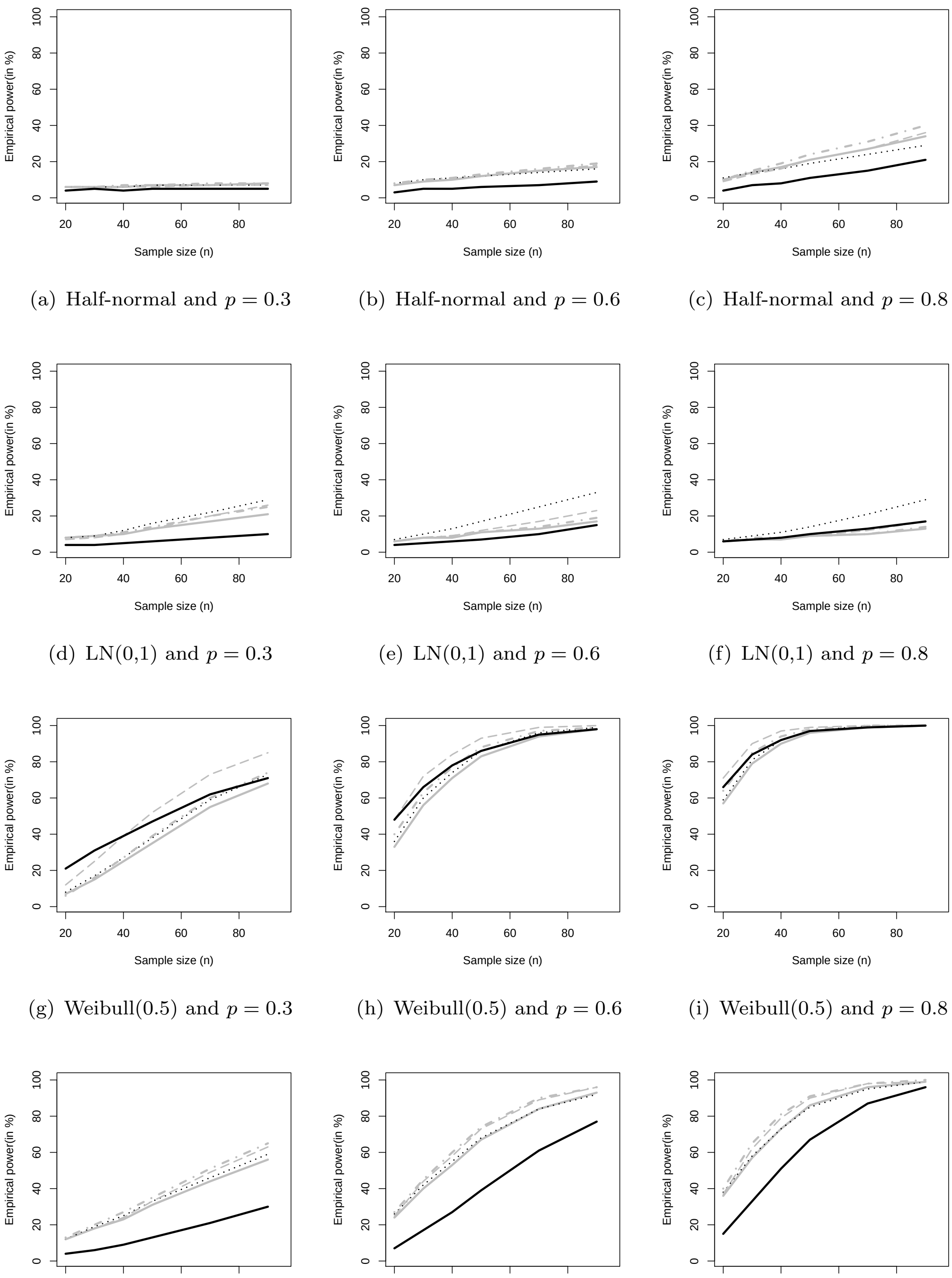
	$n=50$					$n=70$					$n=90$				
Model	$D^\star$	$D^\star_{\rm sp}$	W^2	A^2	U^2	$D^\star$	$D^\star_{\rm sp}$	W^2	A^2	U^2	$D^\star$	$D^\star_{\rm sp}$	W^2	A^2	U^2
Bet(2,1)	89	87	94	92	63	97	96	99	99	84	99	99	100	100	94
HCa(0,1)	6	6	6	6	7	6	5	6	6	6	6	5	6	6	7
HN(0,1)	12	12	13	12	6	15	14	16	15	7	17	16	19	18	9
LN(0,1)	11	17	11	12	7	13	25	14	17	10	17	33	19	23	15
Lom(0.5)	60	53	66	69	64	75	67	81	83	76	85	78	89	90	85
Lom(2)	8	6	9	10	11	10	7	11	12	11	12	8	14	14	13
U(0,1)	23	20	26	22	9	29	24	34	29	12	35	28	41	36	16
Wei(0.5)	83	86	88	93	86	94	96	97	99	95	98	99	99	100	98
Wei(2)	67	68	74	73	39	84	84	90	89	61	93	92	96	96	77
Wei(3)	90	88	93	92	68	97	97	99	99	88	99	99	100	100	96
Wei(6)	97	96	98	98	87	100	99	100	100	97	100	100	100	100	99
$\chi^2(1)$	58	64	64	75	65	75	81	81	89	78	85	90	90	95	87
$\chi^2(3)$	17	22	19	19	8	24	31	28	29	13	30	38	35	37	17
$\chi^2(4)$	35	41	40	40	17	50	57	56	58	29	63	69	70	72	43
Cau(0,1)	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100
Lap(0,1)	100	100	100	100	98	100	100	100	100	100	100	100	100	100	100
N(0,1)	95	95	97	97	83	99	99	100	100	96	100	100	100	100	99
$t(3)$	99	99	100	100	97	100	100	100	100	100	100	100	100	100	100
Exp(0,1)	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5

Table 14. Estimated power (in %) of tests based on the indicated statistics, models and values of n for $p = 0.8$ and $\alpha = 0.05$.

	$n = 20$					$n = 30$					$n = 40$				
Model	$D^\star$	$D^\star_{\rm sp}$	W^2	A^2	U^2	$D^\star$	$D^\star_{\rm sp}$	W^2	A^2	U^2	$D^\star$	$D^\star_{\rm sp}$	W^2	A^2	U^2
Bet(2,1)	69	67	76	72	30	89	86	94	93	60	97	95	99	98	79
HCa(0,1)	13	10	15	15	16	16	12	19	19	20	19	13	23	22	22
HN(0,1)	10	11	10	9	4	14	14	15	13	7	17	16	19	16	8
LN(0,1)	6	7	6	6	6	7	9	8	7	7	7	11	8	8	8
Lom(0.5)	67	64	72	75	73	84	82	89	90	87	93	90	95	96	94
Lom(2)	10	8	12	13	14	14	11	17	18	18	17	12	21	22	20
U(0,1)	23	22	26	22	7	33	29	39	33	13	43	35	51	44	19
Wei(0.5)	57	59	64	71	66	79	81	85	90	84	90	92	94	97	92
Wei(2)	36	38	40	37	15	57	58	65	62	33	73	73	81	79	51
Wei(3)	59	59	65	61	28	82	81	87	86	59	93	92	96	95	79
Wei(6)	77	76	82	79	46	94	92	96	95	79	99	98	99	99	93
$\chi^2(1)$	28	30	34	41	37	45	49	53	62	54	58	63	65	75	65
$\chi^2(3)$	9	11	9	8	4	13	16	14	13	7	17	20	19	18	10
$\chi^2(4)$	16	19	17	15	7	25	30	29	27	14	34	40	40	39	21
Cau(0,1)	96	96	96	96	91	99	99	100	100	99	100	100	100	100	100
Lap(0,1)	88	87	90	89	70	98	97	99	98	94	100	100	100	100	99
N(0,1)	70	70	75	72	40	90	89	93	92	73	97	96	98	98	88
$t(3)$	85	85	88	86	67	97	97	98	98	91	99	99	100	100	98
Exp(0,1)	5	5	5	4	5	5	5	5	5	5	5	5	5	5	5

Table 15. Estimated power (in %) of tests based on the indicated statistics, models and values of n for $p = 0.8$ and $\alpha = 0.05$.

	$n = 50$					$n = 70$					$n = 90$				
Model	$D^\star$	$D^\star_{\rm sp}$	W^2	A^2	U^2	$D^\star$	$D^\star_{\rm sp}$	W^2	A^2	U^2	$D^\star$	$D^\star_{\rm sp}$	W^2	A^2	U^2
Bet(2,1)	99	99	100	100	91	100	100	100	100	99	100	100	100	100	100
HN(0,1)	21	19	24	21	11	27	24	31	27	15	34	29	40	36	21
HCa(0,1)	22	15	26	25	24	28	17	33	30	29	34	20	39	36	35
LN(0,1)	9	14	9	9	10	10	21	10	12	13	13	29	14	17	17
Lom(0.5)	97	96	98	98	97	99	99	100	100	99	100	100	100	100	100
Lom(2)	21	15	25	26	23	27	18	32	33	28	33	23	40	40	34
U(0,1)	53	44	62	55	26	68	54	77	71	39	79	66	87	83	52
Wei(0.5)	96	97	98	99	97	99	100	100	100	99	100	100	100	100	100
Wei(2)	86	85	91	90	67	96	95	98	98	87	99	99	100	99	96
Wei(3)	98	97	99	99	91	100	100	100	100	99	100	100	100	100	100
Wei(6)	100	100	100	100	98	100	100	100	100	100	100	100	100	100	100
$\chi^2(1)$	71	77	77	86	75	86	90	90	95	88	94	96	96	99	94
$\chi^2(3)$	22	27	24	24	13	30	37	34	35	21	39	48	45	48	29
$\chi^2(4)$	45	52	51	52	30	63	68	70	71	48	77	82	84	85	65
Cau(0,1)	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100
Lap(0,1)	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100
N(0,1)	99	99	100	100	96	100	100	100	100	100	100	100	100	100	100
$t(3)$	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100
Exp(0,1)	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5



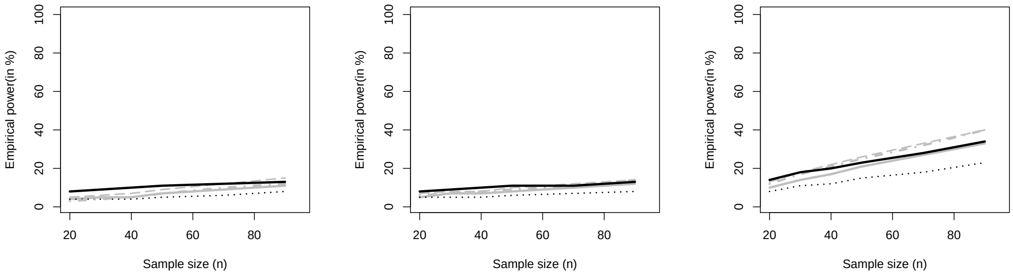
(a) Half-normal and $p = 0.3$ (b) Half-normal and $p = 0.6$ (c) Half-normal and $p = 0.8$

(d) LN(0,1) and $p = 0.3$ (e) LN(0,1) and $p = 0.6$ (f) LN(0,1) and $p = 0.8$

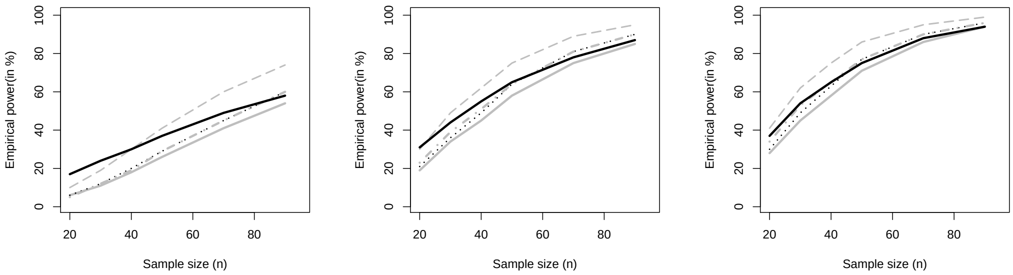
(g) Weibull(0.5) and $p = 0.3$ (h) Weibull(0.5) and $p = 0.6$ (i) Weibull(0.5) and $p = 0.8$

(j) Weibull(2) and $p = 0.3$ (k) Weibull(2) and $p = 0.6$ (l) Weibull(2) and $p = 0.8$

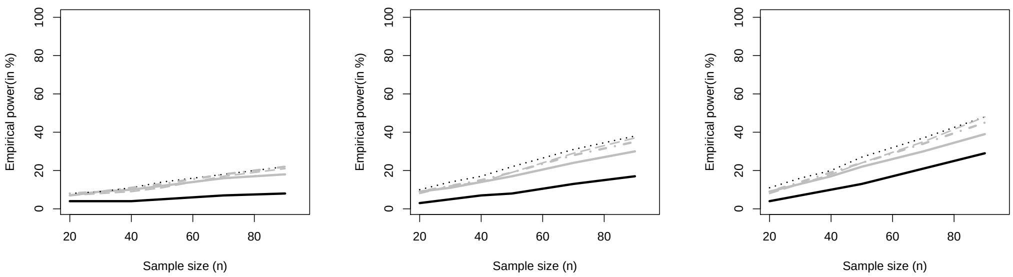
Figure 3. Estimated power of the tests based on $D^\star$ (gray solid line), $D^\star_{\mathrm{SP}}$ (bold dotted line), W^2 (gray dashed and dotted line), A^2 (gray dashed line), and U^2 (bold solid line) for the distribution specified in H_1.



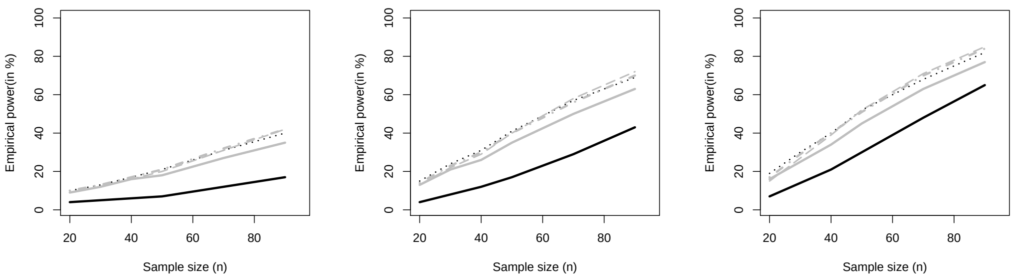
(a) Lomax(2) and $p = 0.3$ (b) Lomax(2) and $p = 0.6$ (c) Lomax(2) and $p = 0.8$



(d) $\chi^2(1)$ and $p = 0.3$ (e) $\chi^2(1)$ and $p = 0.6$ (f) $\chi^2(1)$ and $p = 0.8$



(g) $\chi^2(3)$ and $p = 0.3$ (h) $\chi^2(3)$ and $p = 0.6$ (i) $\chi^2(3)$ and $p = 0.8$



(j) $\chi^2(4)$ and $p = 0.3$ (k) $\chi^2(4)$ and $p = 0.6$ (l) $\chi^2(4)$ and $p = 0.8$

Figure 4. Estimated power of the tests based on $D^{\star}$ (gray solid line), $D^{\star}_{\mathrm{SP}}$ (bold dotted line), W^2 (gray dashed and dotted line), A^2 (gray dashed line), and U^2 (bold solid line) for the distribution specified in H_1.

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